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Analytical solution of the polarized photon transport equation in an infinite uniform medium using cumulant expansion

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An analytical solution for time-dependent polarized photon transport equation in an infinite uniform isotropic medium is studied using a circular representation of the polarized light and expansion in the generalized spherical functions. We extend our cumulant approach for solving the scalar (unpolarized) photon transport equation to the vector (polarized) case. As before, an exact angular distribution is obtained and a cumulant expansion is derived for the polarized photon distribution function. By a cutoff at the second cumulant order, a Gaussian analytical approximate expression of the polarized photon spatial distribution is obtained as a function of the direction of light and time, whose average center position and half-width are always exact. The central limit theorem claims that this spatial distribution approaches accuracy in detail when the number of collisions or time becomes large. The analytical expression of cumulants up to an arbitrary high order is also derived, which can be used for calculating a more accurate polarized photon distribution through a numerical Fourier transform. Contrary to what occurs in other approximation techniques, truncation of the cumulant expansion at order n is exact at that order and cumulants up to and including order n remain unchanged when higher orders are added, at least as applied in our photon transport equation.

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I. INTRODUCTION

Study of the polarized photon transport has lasted for many years since the polarized photon transport equation (PPTE) was formulated by Gans [1] and by Chandrasekhar [2]. Recently, polarization analysis of light migrating in a multiple-scattering medium has been applied to broad fields, such as diagnostics of biological tissues [3–5], atmosphere monitoring [6], and communications. One goal is to develop optical tomography with polarization analysis to enhance ability in image reconstruction of objects inside scattering media. Because of the depolarization effect in a highly scattering medium, scattered photons maintaining polarization are those near ballistic and snake like, which suffer less multiple scattering. Therefore, a tomographic approach using polarized photons will automatically exclude multiple-scattered diffusive photons which blur images. In order to build a proper forward model for tomography using a polarization analysis, a theoretical study of the propagation of polarized light in scattering media becomes practically important.

In polarized photon transport, the intensity of polarized light scattered from a scatterer along a certain direction is determined by many scattering processes at different scattering planes consisting in a ray scattered from the scatterer and rays incident to the scatterer from different directions. In order to properly describe this process, Kuščer and Ribarič [7] employed a circular representation of the polarized components of light and an expansion by generalized spherical functions [8] (or rotation matrices in angular momentum theory [9,10]). The phase matrix, hence, can be analytically expressed by the angular parameters of the incident and scattered rays in fixed coordinates. Based on this formalism, Herman and Lenoble [11] studied the asymptotic behavior of the polarized radiation field at great depths. Domke [12] constructed a system of singular eigenfunctions of the PPTE, for which an integral equation and then a recurrence relation were derived. However, to our knowledge, an explicit analytical expression of the solution of the PPTE has not been obtained. Numerical methods, mainly, Monte Carlo simulations, are the main tools in recent theoretical investigations of light polarization in multiple-scattering media [5,13].

In this paper we derive an analytical solution of the time-dependent PPTE in an infinite uniform medium. Based on our results, inverse image reconstruction of objects inside a scattering medium using polarized light can be developed. The 4×4 phase matrix is assumed to depend only on the scattering angle Θ in the scattering plane: $\mathbf{P}(\mathbf{s},\mathbf{s}_0)=\mathbf{P}(\mathbf{s}\cdot\mathbf{s}_0)=\mathbf{P}(\cos\Theta)$. Under this assumption an arbitrary phase matrix can be handled. By use of the circular representation of polarized light and an expansion in the generalized spherical function [7,9], we extend our approach in solving the scalar (unpolarized) photon transport equation [14,15] using a cumulant expansion to the vector (polarized) case. Terminating at second order, an approximate Gaussian polarized photon spatial distribution is obtained for a given light direction $\mathbf{s}$ as a function of time t. Our solution for the distribution in angle is exact, as are the first and second cumulants in space at any angle and time, which guarantees the correct central position and the correct half-width of the spatial distribution. After many scattering events have taken place, the central limit theorem claims that the spatial Gaussian distribution calculated will become accurate in detail, since all cumulants higher than the second approach small values relative to the appropriate power of the second cumulant. At early times, the spatial distribution is narrow: hence, a distribution function, the mean position and half-width of which are exact, may provide an adequate description of a polarized beam in the presence of noise and finite instrument resolution. An analytical expression of cumulants up to an arbitrary high order is also derived. Using these higher-order cumulants, through a numerical Fourier transform, a more accurate solution of the PPTE can be calculated.

The paper is organized as follows. Section II provides the preliminary formula of the PPTE, the circular representation of polarized light and the generalized spherical function, which have been published in previous literature. In Sec. III, an exact solution of the angular distribution of the polarized light is derived and an expression in the cumulant expansion of the polarized photon distribution is presented. In Sec. IV, by terminating the cumulant expansion at second order a Gaussian approximate spatial distribution as a function of light direction **s** and time t is obtained, and the exact expressions of the first and second cumulants are derived. In Sec. V, an expression of cumulants up to an arbitrary high order is derived. A brief discussion and summary then follows in Sec. VI. In Appendix A, the expressions for coefficients $[B^l_{mn_0}]_i$ in Eq. (15) are presented. In Appendix B, analytical formulas for evaluating integrals in Eq. (42) are derived.

II. CIRCULAR REPRESENTATION AND GENERALIZED SPHERICAL FUNCTIONS

In this section, we summarize the description of polarized light propagation in a scattering medium discussed in the previous literature.

Considering a light beam traveling along a direction **s**, we choose a reference plane through the direction of propagation. Two complex components of the electric field **E**, such as $E_\parallel = a_1 \exp(i\delta_1)$, the component parallel to the reference plane, and $E_\perp = a_2 \exp(i\delta_2)$, the component perpendicular to the reference plane, can be used to describe a single coherent beam. Four real components were introduced by Stokes [16], each with the dimension of the square of a field or, more precisely, an intensity. The four Stokes parameters are collected into a four-element array $\mathbf{I}^{SP} = [I, Q, U, V]$ [17]. The component I is the total intensity:

$$I = \langle a_1^2 \rangle + \langle a_2^2 \rangle = \langle |E_\parallel|^2 + |E_\perp|^2 \rangle. \tag{1a}$$

The component Q describes a linear polarization:

$$Q = \langle a_1^2 \rangle - \langle a_2^2 \rangle = \langle |E_\parallel|^2 - |E_\perp|^2 \rangle. \tag{1b}$$

The component U describes a linear polarization 45° relative to the reference plane:

$$U = \langle 2a_1 a_2 \cos\delta \rangle = \langle |E(45°)|^2 - |E(-45°)|^2 \rangle, \tag{1c}$$

where

$$E(\pm 45°) \equiv 2^{-1/2}(E_\parallel \pm E_\perp).$$

The last component V is the difference between the intensities of right- and left-circularly polarized light:

$$V = \langle 2a_1 a_2 \sin\delta \rangle = \langle |E_R|^2 - |E_L|^2 \rangle, \tag{1d}$$

where the right- and left-circular components of field are

$$E_L^R \equiv 2^{-1/2}(E_\parallel \mp iE_\perp).$$

In Eq. (1), $\delta = \delta_1 - \delta_2$; the angular brackets mean the average over many waves with independent phases in a light beam.

To give further physical meaning to the symbols, we note that the use of the Cauchy-Schwarz inequality leads to the inequality

$$I^2 \geq Q^2 + U^2 + V^2. \tag{2}$$

For a coherent beam, which requires no averages in Eqs. 1(a)–1(d), the equality is automatically obeyed. The opposite extreme case is unpolarized light for which

$$Q = U = V = 0,$$

and the total intensity I is totally incoherent. More generally, the difference between the left- and right-hand sides of the inequality, Eq. (2), constitutes the incoherent part of the total intensity.

The kinetic equation for a polarized photon distribution function $\mathbf{I}(\mathbf{r},\mathbf{s},t)$ as a function of time t, position **r**, and direction **s**, in an infinite uniform medium, from a point pulse polarized light source, $\mathbf{I}^{(0)}\delta(\mathbf{r}-\mathbf{r}_0)\delta(\mathbf{s}-\mathbf{s}_0)\delta(t-0)$, is given by [2]

$$\partial \mathbf{I}(\mathbf{r},\mathbf{s},t)/\partial t + c\mathbf{s}\cdot\nabla_r \mathbf{I}(\mathbf{r},\mathbf{s},t) + \mu_a \mathbf{I}(\mathbf{r},\mathbf{s},t) = \mu_s \int \mathbf{P}(\mathbf{s},\mathbf{s}')[\mathbf{I}(\mathbf{r},\mathbf{s}',t) - \mathbf{I}(\mathbf{r},\mathbf{s},t)]d\mathbf{s}' + \mathbf{I}^{(0)}\delta(\mathbf{r}-\mathbf{r}_0)\delta(\mathbf{s}-\mathbf{s}_0)\delta(t-0). \tag{3}$$

The quantities in the Stokes parameter (SP) representation will be marked by adding a superindex, for example, $\mathbf{I}^{SP}$. With a rotation of the reference plane through an angle $\alpha \geq 0$ (in the counterclockwise direction, when looking in the direction of propagation) around the light propagation direction, **I** varies as $\mathbf{I}' = \mathbf{L}(\alpha)\mathbf{I}$. In the SP representation, the relation is given by

$$\begin{bmatrix} I' \\ Q' \\ U' \\ V' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos 2\alpha & \sin 2\alpha & 0 \\ 0 & -\sin 2\alpha & \cos 2\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} I \\ Q \\ U \\ V \end{bmatrix}. \tag{4}$$

Usually, a meridian plane containing the z axis and the light direction **s** is used as the reference plane for the description of the polarization state [2,9] as shown in Fig. 1. In Eq. (3), c is the light speed in the medium, μ_s is the scattering rate (per unit time), μ_a is the absorption rate, and $\mathbf{P}(\mathbf{s},\mathbf{s}')$ is a 4×4 phase matrix. The following form of the 4×4 phase matrix [9] is used:

$$\mathbf{P}(\mathbf{s},\mathbf{s}') = \mathbf{L}(\pi-\chi)\mathbf{P}(\cos\Theta)\mathbf{L}(-\chi'), \tag{5}$$

where Θ is the angle between light rays before and after scattering, and the matrices $\mathbf{L}(-\chi')$ and $\mathbf{L}(\pi-\chi)$ are those required to rotate meridian planes before and after scattering

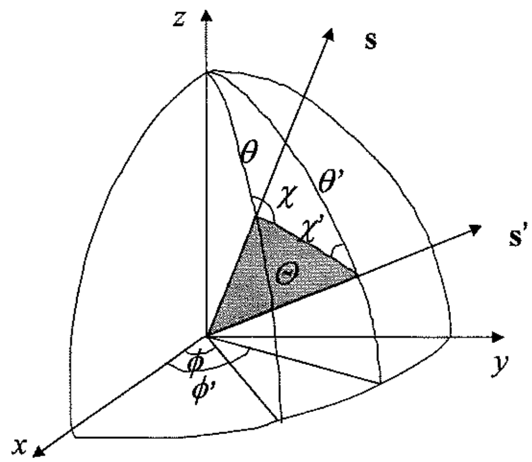

FIG. 1. Geometry of the scattering plane and the reference planes related to the incident ray, $\mathbf{s}'(\theta',\phi')$, and the scattered ray, $\mathbf{s}(\theta,\phi)$. The dark plane is the scattering plane. χ is the angle between the meridian plane $(\mathbf{s},\mathbf{z})$ and the scattering plane. χ' is the angle between the meridian plane $(\mathbf{s}',\mathbf{z})$ and the scattering plane.

onto or from a local scattering plane, as shown in Fig. 1. The intrinsic property of scattering mechanism is described by the 4×4 scattering function $\mathbf{P}(\cos\Theta)$, which involves $\cos\Theta=\mathbf{s}\cdot\mathbf{s}'$.

It is convenient to use a representation of the polarized light in which $\mathbf{L}(\alpha)$ is diagonal, rather than Eq. (4). A circular parameter representation (CP) was first proposed by Kuščer and Ribarič [7]. Later, a more precise definition of the CP, which matches with the initial definition of polarized light in the SP representation by Chandrasekhar [2], was presented by Hovenier and van der Mee [9]. Hereafter we use the definition of the CP in Ref. [9], which is given by $\mathbf{I}^{\text{CP}}=[I_2,I_0,I_{-0},I_{-2}]$, where $I_0=(I+V)/2$, $I_{-0}=(I-V)/2$, $I_2=(Q+iU)/2$, and $I_{-2}=(Q-iU)/2$, or $\mathbf{I}^{\text{CP}}=\mathbf{T}\mathbf{I}^{\text{SP}}$, with

$$\mathbf{T}=\frac{1}{2}\begin{bmatrix}0 & 1 & i & 0\\ 1 & 0 & 0 & 1\\ 1 & 0 & 0 & -1\\ 0 & 1 & -i & 0\end{bmatrix}. \tag{6}$$

In the CP, a rotation of the reference plane through an angle α around the light direction causes I_m to be multiplied by $\exp(-im\alpha)$. Notice that I_0 and I_{-0} actually have the same rotational property. For the phase matrix, the transform $\mathbf{P}^{\text{CP}}=\mathbf{T}\mathbf{P}^{\text{SP}}\mathbf{T}^{-1}$ is given between two representations.

In the CP, it is convenient to expand the phase matrix $\mathbf{P}^{\text{CP}}$ using generalized spherical functions (GSF's). The generalized spherical functions, which are related to irreducible representations of the rotation group on three nonzero Euler's angles, are defined as follows [8].

For $l\geq \sup(|m|,|n|)$ and $\mu=\cos\theta$,

$$P^l_{m,n}(\mu)=A^l_{m,n}(1-\mu)^{-(n-m)/2}(1+\mu)^{-(n+m)/2}\times\frac{d^{l-n}}{d\mu^{l-n}}[(1-\mu)^{l-m}(1+\mu)^{l+m}], \tag{7}$$

with

$$A^l_{m,n}=\frac{(-1)^{l-m}(i)^{n-m}}{2^l(l-m)!}\left[\frac{(l-m)!(l+n)!}{(l+m)!(l-n)!}\right]^{1/2}.$$

This function is directly related [9] to the rotation matrix $d^l_{mn}(\theta)$ in angular momentum theory [10] by $d^l_{mn}(\theta)=(i)^{n-m}P^l_{mn}(\cos\theta)$. Some symmetry properties of $P^l_{m,n}(\mu)$ are $P^l_{m,n}(\mu)=P^l_{n,m}(\mu)=P^l_{-m,-n}(\mu)$. The orthogonality relation for $P^l_{m,n}(\mu)$ is given by [8,9]

$$(-1)^{m+n}\int_{-1}^{1}P^l_{m,n}(\mu)P^{l'}_{m,n}(\mu)d\mu=\frac{2}{2l+1}\delta_{l,l'}. \tag{8}$$

The phase matrix in the CP can be expressed using the generalized spherical functions [7,9]. For notational simplicity, in the following the quantities without a superindex are understood to be in the CP. Denoting $\mathbf{s}=(\mu,\phi)$ and $\mathbf{s}'=(\mu',\phi')$, the addition theorem of GSF's [9] is given by (see Fig. 1)

$$\exp(im\chi)P^l_{m,n}(\cos\Theta)\exp(in\chi')=(-1)^{m+n}\sum_{s=-l}^{l}(-1)^sP^l_{m,s}(\mu)P^l_{s,n}(\mu')\times\exp[-is(\phi-\phi')]. \tag{9}$$

Using this addition theorem of GSF's, the variables χ, χ', and Θ in Eq. (5) can be eliminated, and the components of the phase matrix in the CP can be expressed using the angular parameters of the incident and scattered ray in fixed coordinates. If we expand elements of the CP phase matrix in the scattering plane, $P_{mn}(\cos\Theta)$, by GSF's,

$$P_{mn}(\cos\Theta)=\frac{1}{4\pi}\sum_l p^l_{mn}P^l_{m,n}(\cos\Theta),$$

then, using Eq. (9), the 4×4 phase matrix in fixed coordinates can be written as

$$P_{mn}(\mu,\phi;\mu',\phi')=\frac{1}{4\pi}\sum_l p^l_{mn}\sum_{s=-l}^{l}(-1)^sP^l_{m,s}(\mu)P^l_{s,n}(\mu')\times\exp[-is(\phi-\phi')], \tag{10}$$

with indices m, $n=2,0,-0,-2$ and $l\geq\sup(|m|,|n|)$.

The coefficients p^l_{mn} provide an intrinsic description of the scattering mechanism. In most useful cases, the coefficients p^l_{mn} have the properties [7,9] that (i) p^l_{mm} and p^l_{m-m} are real, (ii) $p^l_{mn}=p^l_{nm}=p^l_{-m-n}$, and (iii) $p^l_{20}=[p^l_{2-0}]^*$ (the asterisk means complex conjugate). Therefore, for each $l\geq 2$, there are six independent real elements p^l_{00}, p^l_{22}, p^l_{0-0}, p^l_{2-2}, $\text{Re}[p^l_{20}]$, $\text{Im}[p^l_{20}]$. For $l=0$ or 1, only p^l_{00} and p^l_{0-0} are nonzero. These numerical coefficients were calculated using Mie theory for some examples by De Rooij and van der Stap [18]. These p^l_{mn}, together with μ_s and μ_a, are parameters that describe the nature of the scattering process and are treated as known in our solution of the transport equation.

III. DERIVATION

Having the above knowledge, we analytically solve Eq. (3) in an infinite uniform isotropic medium. Using a procedure similar to that discussed in Refs. [14] and [15], we first study the dynamics of the photon distribution in the light direction space in the CP, $\mathbf{F}(\mathbf{s},\mathbf{s}_0,t)$, which is a vector of four components, on a spherical surface for $\mathbf{s}$ of radius 1. The kinetic equation for $\mathbf{F}(\mathbf{s},\mathbf{s}_0,t)$ can be obtained by integrating Eq. (3) over whole space $\mathbf{r}$. The spatial independence of μ_s, μ_a, and $\mathbf{P}(\mathbf{s},\mathbf{s}')$ retains translation invariance. Thus the integral of Eq. (3) obeys

$$\partial\mathbf{F}(\mathbf{s},\mathbf{s}_0,t)/\partial t+\mu_a\mathbf{F}(\mathbf{s},\mathbf{s}_0,t)+\mu_s\left[\mathbf{F}(\mathbf{s},\mathbf{s}_0,t)-\int\mathbf{P}(\mathbf{s},\mathbf{s}')\mathbf{F}(\mathbf{s}',\mathbf{s}_0,t)d\mathbf{s}'\right]=\mathbf{I}^{(0)}\delta(\mathbf{s}-\mathbf{s}_0)\delta(t-0). \tag{11}$$

Since the integral of the gradient term over all space vanishes, as shown in Ref. [14], if we expand $\mathbf{F}(\mathbf{s},\mathbf{s}_0,t)$ in GSF's, its l components should not be coupled to each other. The mth component of $\mathbf{F}(\mathbf{s},\mathbf{s}_0,t)$, with the initial polarization in unit n_0 state, can be expanded in GSF's in the following form:

$$F_{mn_0}(\mathbf{s},\mathbf{s}_0,t)=\sum_l F^l_{mn_0}(t)\sum_s(-1)^s P^l_{m,s}(\mu)P^l_{s,n_0}(\mu_0)\times\exp[-is(\phi-\phi_0)]\exp(-\mu_a t), \tag{12}$$

with m, $n_0=2,0,-0,-2$, $l\geqslant\sup(|m|,|n|)$. When $\mathbf{s}_0$ is set along the z direction and the initial reference plane is set as the x-o-z plane, Eq. (12) specializes to

$$F_{mn_0}(\mathbf{s},\hat{\mathbf{z}},t)=\sum_l F^l_{mn_0}(t)P^l_{m,n_0}(\mu)\exp(-in_0\phi)\exp(-\mu_a t). \tag{13}$$

Substituting Eq. (12) [or Eq. (13)] into Eq. (11), using the expression, Eq. (10), of the phase matrix, and the orthogonality relation of GSF's, Eq. (8), an analytically solvable equation for $F^l_{mn_0}(t)$ for each l is obtained:

$$dF^l_{mn_0}(t)/dt=\sum_n \Pi^l_{mn}F^l_{nn_0}(t), \tag{14a}$$

with $\Pi^l_{mn}=\mu_s[\delta_{m,n}-p^l_{mn}/(2l+1)]$. The initial condition $F_m(\mathbf{s},\mathbf{s}_0,t=0)=\delta_{m,n_0}\delta(\mathbf{s}-\mathbf{s}_0)$ and the orthogonality relation, Eq. (8), lead to

$$F^l_{mn_0}(t=0)=\delta_{m,n_0}(2l+1)/4\pi. \tag{14b}$$

The solution of Eq. (14) can be expanded in terms of eigenstates:

$$F^l_{mn_0}(t)=\frac{2l+1}{4\pi}\sum_i[B^l_{mn_0}]_i\exp(-\lambda^l_i t),\quad i=1,2,3,4, \tag{15}$$

with the eigenvalues given by

$$\lambda^l_i=(1/2)\left\{(\Pi^l_{00}+\Pi^l_{22}\pm\Pi^l_{0-0}\pm\Pi^l_{2-2})+\left[(\Pi^l_{00}-\Pi^l_{22}\pm\Pi^l_{0-0}\mp\Pi^l_{2-2})^2+16\left\{\begin{matrix}+\mathrm{Re}(\Pi_{20})\\-\mathrm{Im}(\Pi_{20})\end{matrix}\right\}^2\right]^{1/2}\right\}, \tag{16}$$

for $i=1,2$, and for $i=3,4$, the sign$+$before square brackets in Eq. (16) is replaced by $-$. The constant coefficients $[B^l_{mn_0}]_i$ can be analytically determined using standard linear algebra from the initial condition, Eq. (14b). A detailed expression for $[B^l_{mn_0}]_i$ is presented in Appendix A.

Equation (12) [or Eq. (13)], combined with Eqs. (15) and (16) and the coefficients $[B^l_{mn_0}]_i$ in Appendix A, provides an exact CP solution in the light direction space. In the SP representation, we have

$$\mathbf{F}^{\mathrm{SP}}(\mathbf{s},\mathbf{s}_0,t)=\mathbf{T}^{-1}\mathbf{F}(\mathbf{s},\mathbf{s}_0,t)\mathbf{T}. \tag{17}$$

It can be proved that all components of $\mathbf{F}^{\mathrm{SP}}(\mathbf{s},\mathbf{s}_0,t)$ are real numbers. The mth component $[m=I,Q,U,V]$ of the angular distribution function in the SP representation, with the initial polarized state $\mathbf{I}^{\mathrm{SP}(0)}$, is obtained by

$$F^{\mathrm{SP}}_m(\mathbf{s},\mathbf{s}_0,t)=[\mathbf{F}^{\mathrm{SP}}(\mathbf{s},\mathbf{s}_0,t)\mathbf{I}^{\mathrm{SP}(0)}]_m. \tag{18}$$

Equation (17) serves as the exact Green's function of polarized light propagation in the light direction space. Since in an infinite uniform medium this function is independent of the source position $\mathbf{r}_0$, requirements for a Green's function are satisfied: especially, the Chapman-Kolmogorov condition is obeyed: $\int d\mathbf{s}'\,\mathbf{F}^{\mathrm{SP}}(\mathbf{s}'',\mathbf{s}',t-t')\mathbf{F}^{\mathrm{SP}}(\mathbf{s}',\mathbf{s},t'-t_0)=\mathbf{F}^{\mathrm{SP}}(\mathbf{s}'',\mathbf{s},t-t_0)$. In fact, in an infinite uniform medium, this propagator determines all time evolution of polarized light, including its spatial distribution, because displacement is an integration of velocity, $c\mathbf{s}(t)$, over time. The mth component $[m=I,Q,U,V]$ of the photon distribution function in the SP representation, $I^{\mathrm{SP}}_m(\mathbf{r},\mathbf{s},t)$, with the source located at $\mathbf{r}_0=0$, the initial direction $\mathbf{s}_0$, and the initial polarization $\mathbf{I}^{\mathrm{SP}(0)}$, is given by

$$I^{\mathrm{SP}}_m(\mathbf{r},\mathbf{s},t)=\left\langle\delta\left(\mathbf{r}-c\int_0^t\mathbf{s}(t')dt'\right)\delta(\mathbf{s}(t)-\mathbf{s})\right\rangle^{\mathrm{SP}}_m, \tag{19}$$

where $\langle\cdots\rangle^{\mathrm{SP}}_m$ means the mth component of the ensemble average in the light direction space in the SP representation. The first δ function ensures that the displacement $\mathbf{r}-0$ is given by a path integral. The second δ function assures the correct final value of the direction. Equation (19) is a formally exact solution, but cannot be evaluated directly. We make a Fourier transform for the first δ function in Eq. (19), then make a cumulant expansion [19], and obtain

$$I_m^{\rm SP}(\mathbf{r},\mathbf{s},t)=F_m^{\rm SP}(\mathbf{s},\mathbf{s}_0,t)\frac{1}{(2\pi)^3}\int d\mathbf{q}\exp\left\{i\mathbf{q}\cdot\mathbf{r}+\sum_{k=1}^{\infty}\frac{(-ic)^k}{k!}\sum_{j_k}\cdots\sum_{j_1}q_{j_k}\cdots q_{j_1}\times\left[\left\langle\int_0^t dt_k\cdots\int_0^t dt_1\, T[s_{j_k}(t_k)\cdots s_{j_1}(t_1)]\right\rangle_c\right]_m^{\rm SP}\right\},\tag{20}$$

where T denotes time-ordered multiplication [20] and $F_m^{\rm SP}(\mathbf{s},\mathbf{s}_0,t)$ is given by Eq. (18). In Eq. (20) the index c denotes a cumulant, which is defined in textbooks of statistics [21] and statistical physics [19]. As for an arbitrary random variable A, we have $\langle A\rangle_c=\langle A\rangle$, $\langle A^2\rangle_c=\langle A^2\rangle-\langle A\rangle\langle A\rangle$, and a general expression relating $\langle A^i\rangle$ and $\langle A^i\rangle_c$, which is given by

$$\langle A^i\rangle=i!\sum_{i_1,i_2,\ldots}\frac{1}{i_1!}\left(\frac{\langle A\rangle}{1!}\right)^{i_1}\frac{1}{i_2!}\left(\frac{\langle A^2\rangle_c}{2!}\right)^{i_2}\cdots\frac{1}{i_n!}\left(\frac{\langle A^n\rangle_c}{n!}\right)^{i_n}\cdots\delta(i-i_1-2i_2-\cdots-ni_n-\cdots),\tag{21}$$

Hence, if $\langle A^i\rangle$, $i=1,2,\ldots,k$, have been calculated, $\langle A^i\rangle_c$, $i=1,2,\ldots,k$, can be recursively obtained and conversely [19]. The kth moment (the term without index c) which, according to the cumulant expansion theorem, is related to $\int d\mathbf{r}\, r_{j_k}\cdots r_{j_1}I_m^{\rm SP}(\mathbf{r},\mathbf{s},t)$. This moment can be evaluated using a standard time-dependent Green's function approach, which is given by

$$\left[\left\langle\int_0^t dt_k\cdots\int_0^t dt_1\, T[s_{j_k}(t_k)\cdots s_{j_1}(t_1)]\right\rangle\right]_m^{\rm SP}=\frac{1}{F_m^{\rm SP}(\mathbf{s},\mathbf{s}_0,t)}\left\{\left[\int_0^t dt_k\int_0^{t_k}dt_{k-1}\cdots\int_0^{t_2}dt_1\int d\mathbf{s}^{(k)}\int d\mathbf{s}^{(k-1)}\cdots\int d\mathbf{s}^{(1)}\mathbf{F}^{\rm SP}(\mathbf{s},\mathbf{s}^{(k)},t-t_k)s_{j_k}^{(k)}\times\mathbf{F}^{\rm SP}(\mathbf{s}^{(k)},\mathbf{s}^{(k-1)},t_k-t_{k-1})s_{j_{k-1}}^{(k-1)}\cdots\mathbf{F}^{\rm SP}(\mathbf{s}^{(2)},\mathbf{s}^{(1)},t_2-t_1)s_{j_1}^{(1)}\mathbf{F}^{\rm SP}(\mathbf{s}^{(1)},\mathbf{s}_0,t_1-0)\mathbf{I}^{\rm SP(0)}\right]_m+(\text{perm.})\right\},\tag{22}$$

where the abbreviation "perm" means all $k!-1$ terms obtained by permutation of $\{j_i\}$, $i=1,\ldots,k$, from the first term.

In Eq. (22), $\mathbf{F}^{\rm SP}(\mathbf{s}^{(i)},\mathbf{s}^{(i-1)},t_i-t_{i-1})$ is given by Eq. (17). Since Eq. (22) is obtained using a Green's function approach without making any approximation and Eq. (17) is an exact expression of the angular Green's function, Eq. (22) provides an exact formula for the kth moment. If we are able to exactly evaluate Eq. (22) up to kth order, through Eq. (21), we can obtain the exact cumulants of the distribution up to the kth order.

IV. GAUSSIAN APPROXIMATION OF THE DISTRIBUTION

Terminating Eq. (20) at second order of the cumulant and setting $\mathbf{s}$ in Cartesian coordinates, integration over $\mathbf{q}$ in Eq. (20) can be analytically performed, which leads to the following Gaussian approximation expression of the polarized photon distribution. When the initial $\mathbf{s}_0$ is set along $\mathbf{z}$, it is given by

$$I_m^{\rm SP}(\mathbf{r},\mathbf{s},t)=\frac{F_m^{\rm SP}(\mathbf{s},\hat{\mathbf{z}},t)}{(4\pi)^{3/2}}\frac{1}{[\det D_m^{\rm SP}]^{1/2}}\exp\left[-\frac{1}{4}[(D_m^{\rm SP})^{-1}]_{\alpha\beta}\times(r_\alpha-\langle R_\alpha\rangle_m^{\rm SP})(r_\beta-\langle R_\beta\rangle_m^{\rm SP})\right],\tag{23}$$

with $m=I,Q,U,V$ and α, $\beta=x,y,z$. In Eq. (23), $\langle R_\alpha\rangle_m^{\rm SP}$ represents the position of the average center of the distribution, and $[D_m^{\rm SP}]_{\alpha\beta}$ is related to the half-width of the spread of the distribution, which is given by

$$[D_m^{\rm SP}]_{\alpha\beta}=[\langle R_\alpha R_\beta\rangle_m^{\rm SP}-\langle R_\alpha\rangle_m^{\rm SP}\langle R_\beta\rangle_m^{\rm SP}]/2.\tag{24}$$

$\langle R_\alpha\rangle_m^{\rm SP}$ in Eq. (23) and $\langle R_\alpha R_\beta\rangle_m^{\rm SP}$ in Eq. (24) can be evaluated using, separately, the first order and the second order of Eq. (22):

$$\langle R_\alpha\rangle_m^{\rm SP}\equiv c\left\langle\int_0^t dt' s_\alpha(t')\right\rangle_m^{\rm SP}=\frac{c}{F_m^{\rm SP}(\mathbf{s},\hat{\mathbf{z}},t)}\left[\int_0^t dt'\int d\mathbf{s}'\mathbf{F}^{\rm SP}(\mathbf{s},\mathbf{s}',t-t')s'_\alpha\times\mathbf{F}^{\rm SP}(\mathbf{s}',\hat{\mathbf{z}},t')\mathbf{I}^{\rm SP(0)}\right]_m\tag{25}$$

and

$$\langle R_\alpha R_\beta \rangle_m^{\rm SP} \equiv c^2 \left\langle \int_0^t dt' \int_0^t dt'' T[s_\alpha(t') s_\beta(t'')] \right\rangle_m^{\rm SP} = \frac{c^2}{F_m^{\rm SP}(\mathbf{s},\hat{\mathbf{z}},t)} \times \left\{ \left[\int_0^t dt' \int_0^{t'} dt'' \int d\mathbf{s}' \int d\mathbf{s}'' \, \mathbf{F}^{\rm SP}(\mathbf{s},\mathbf{s}',t-t') s'_\alpha \mathbf{F}^{\rm SP}(\mathbf{s}',\mathbf{s}'',t'-t'') s''_\beta \mathbf{F}^{\rm SP}(\mathbf{s}'',\hat{\mathbf{z}},t'') \mathbf{I}^{\rm SP(0)} \right]_m + (\text{t.c.}) \right\}, \quad (26)$$

where (t.c.) means that the second term is obtained by exchanging the indices α and β in the first term. As discussed at end of last section, Eqs. (25) and (26) provide exact expressions for evaluation of the first and the second moments.

In evaluation of Eqs. (25) and (26), it is convenient to use the components of $\mathbf{s}$ in a spherical harmonic basis:

$$\mathbf{s} = [s_1, s_0, s_{-1}] = [-2^{-1/2} \sin\theta e^{+i\phi}, \cos\theta, +2^{-1/2} \sin\theta e^{-i\phi}], \quad (27)$$

and first calculate the corresponding quantities in the CP. Hence we write Eq. (25) as

$$\langle R_\alpha \rangle_m^{\rm SP} = \frac{1}{F_m^{\rm SP}(\mathbf{s},\hat{\mathbf{z}},t)} \left[\sum_j U_{\alpha j} \mathbf{T}^{-1} \langle R_j \rangle \mathbf{T} \mathbf{I}^{\rm SP(0)} \right]_m, \quad (28)$$

with $\alpha = x,y,z$ and $j = 1,0,-1$, the indices of the spherical harmonic basis. U is a matrix for the transform from a spherical harmonic basis to a Cartesian basis, $s_\alpha = U_{\alpha j} s_j$, given by

$$U = \begin{bmatrix} -2^{-1/2} & 0 & 2^{-1/2} \\ 2^{-1/2} i & 0 & 2^{-1/2} i \\ 0 & 1 & 0 \end{bmatrix}. \quad (29)$$

$\langle \mathbf{R} \rangle$ in Eq. (28) is defined in the CP as

$$\langle R_j \rangle_{mn_0} \equiv c \int_0^t dt' \int d\mathbf{s}' \sum_n F_{mn}(\mathbf{s},\mathbf{s}',t-t') s'_j F_{mn_0}(\mathbf{s}',\hat{\mathbf{z}},t'), \quad (30)$$

where $F_{mn}(\mathbf{s}_2,\mathbf{s}_1,t_2-t_1)$ is the exact angular Green's function in the CP, Eq. (12). Similarly, Eq. (26) is written as

$$\langle R_\alpha R_\beta \rangle_m^{\rm SP} = \frac{1}{F_m^{\rm SP}(\mathbf{s},\hat{\mathbf{z}},t)} \left[\sum_{j_1} \sum_{j_2} (U_{\alpha j_1} U_{\beta j_2} + U_{\alpha j_2} U_{\beta j_1}) \mathbf{T}^{-1} \langle \mathbf{R}_{j_2} \mathbf{R}_{j_1} \rangle \mathbf{T} \mathbf{I}^{\rm SP(0)} \right]_m, \quad (31)$$

where $\langle \mathbf{R}_{j_2} \mathbf{R}_{j_1} \rangle$ is defined in the CP as

$$\langle R_{j_2} R_{j_1} \rangle_{mn_0} \equiv c^2 \int_0^t dt' \int_0^{t'} dt'' \int d\mathbf{s}' \int d\mathbf{s}'' \times \sum_{n_2} F_{mn_2}(\mathbf{s},\mathbf{s}',t-t') s'_{j_2} \times \sum_{n_1} F_{n_2 n_1}(\mathbf{s}',\mathbf{s}'',t'-t'') s''_{j_1} F_{n_1 n_0}(\mathbf{s}'',\hat{\mathbf{z}},t''), \quad (32)$$

where j_1 and j_2 are spherical components, 1, 0, -1.

In the evaluation of Eqs. (30) and (32), a recurrence relation of GSP's is used, which is directly derived from angular momentum theory [10]. Defining $s_j = \mu_j e^{ij\phi}$, with $j = 1,0,-1$, we have

$$\mu_j P_{m,n}^l(\cos\theta) = \gamma_j \sum_h \langle l,1,m,0 | l+h,m \rangle \times \langle l,1,n,\pm j | l+h, n\pm j \rangle P_{m,n\pm j}^{l+h}(\cos\theta), \quad j,h = +1,0,-1, \quad (33)$$

with $\gamma_{\mp 1} = \frac{-}{+} i$ and $\gamma_0 = 1$, and $\langle l_1,l_2,m_1,m_2 | L,M \rangle$ are the Clebsch-Gordan coefficients in angular momentum theory [10], given by

$$\langle l-h,1,m,-j | l,m-j \rangle = \begin{bmatrix} \left[\frac{(l-m)(l-m+1)}{(2l-1)2l}\right]^{1/2} & \left[\frac{(l+m)(l-m+1)}{2l(l+1)}\right]^{1/2} & \left[\frac{(l+m)(l+m+1)}{(2l+2)(2l+3)}\right]^{1/2} \\ \left[\frac{(l-m)(l+m)}{(2l-1)l}\right]^{1/2} & \left[\frac{m^2}{l(l+1)}\right]^{1/2} & -\left[\frac{(l+m+1)(l-m+1)}{(l+1)(2l+3)}\right]^{1/2} \\ \left[\frac{(l+m)(l+m+1)}{(2l-1)2l}\right]^{1/2} & -\left[\frac{(l-m)(l+m+1)}{2l(l+1)}\right]^{1/2} & \left[\frac{(l-m)(l-m+1)}{(2l+2)(2l+3)}\right]^{1/2} \end{bmatrix}, \quad (34)$$

with the row index (from above) $j = 1,0,-1$ and the column index (from left) $h = 1,0,-1$. These recurrence relations of GSF's were provided in Ref. [8] with some misprints. Substituting Eqs. (12) and (13) into Eq. (30), and using Eq. (33) and the orthogonality relation of GSF's, Eq. (8), integrations over ds' and dt' in Eq. (30) can be analytically performed. When the final direction $\mathbf{s}=(\theta,\phi)$, we have

$$\langle R_j\rangle_{mn_0}=c\sum_l P^l_{m,n_0-j}(\cos\theta)e^{-i(n_0-j)\phi}\gamma_j \times\sum_n\sum_h \frac{2(l-h)+1}{4\pi}D^{l,h}_{m,n,n_0}(t)\langle l-h,1,n,0|l,n\rangle\langle l-h,1,n_0,-j|l,n_0-j\rangle, \quad (35)$$

with $n=2,0,-0,-2$, $h=+1,0,-1$ and

$$D^{l,h}_{m,n,n_0}(t)=\sum_{ij}[B^l_{mn}]_i[B^{l-h}_{nn_0}]_j\left[\frac{\exp(-\lambda^{l-h}_j t)-\exp(-\lambda^l_i t)}{\lambda^l_i-\lambda^{l-h}_j}\right]\exp(-\mu_a t), \quad i,j=1,2,3,4. \quad (36)$$

Similarly, integrations in Eq. (32) can also be analytically performed. We have

$$\langle R_{j_2}R_{j_1}\rangle_{mn_0}=c^2\sum_l P^l_{m,n_0-j_2-j_1}(\cos\theta)e^{-i(n_0-j_2-j_1)\phi}\gamma_{j_2}\gamma_{j_1}\sum_{n_2}\sum_{n_1}\sum_{h_2}\sum_{h_1}\frac{2(l-h_2-h_1)+1}{4\pi} \times E^{l,h_2,h_1}_{m,n_2,n_1,n_0}(t)\langle l-h_2,1,n_2,0|l,n_2\rangle\langle l-h_2,1,n_0-j_1,-j_2|l,n_0-j_1-j_2\rangle \times\langle l-h_2-h_1,1,n_1,0|l-h_2,n_1\rangle\langle l-h_2-h_1,1,n_0,-j_1|l-h_2,n_0-j_1\rangle, \quad (37)$$

with n_1, $n_2=2,0,-0,-2$, $h_1,h_2=+1,0,-1$, and

$$E^{l,h_2,h_1}_{m,n_2,n_1,n_0}(t)=\sum_{ijf}[B^l_{mn_2}]_i[B^{l-h_2}_{n_2n_1}]_j[B^{l-h_2-h_1}_{n_1n_0}]_f \times\left[\frac{\exp(-\lambda^{l-h_2-h_1}_f t)-\exp(-\lambda^l_i t)}{(\lambda^{l-h_2}_j-\lambda^{l-h_2-h_1}_f)(\lambda^l_i-\lambda^{l-h_2-h_1}_f)}-\frac{\exp(-\lambda^{l-h_2}_j t)-\exp(-\lambda^l_i t)}{(\lambda^{l-h_2}_j-\lambda^{l-h_2-h_1}_f)(\lambda^l_i-\lambda^{l-h_2}_j)}\right]\exp(-\mu_a t). \quad (38)$$

Up to now, algebraic analytical expressions for the first cumulant (the average center of the distribution) and the second cumulant (the half-width of spread) have been derived. Equations (36) and (38) involve the related scattering parameters: μ_s and Π^l_{mn} [defined after Eq. (14)], through λ^l_i in Eq. (16) and $[B^l_{mn_0}]_i$ in Appendix A, and the absorption parameter, μ_a. Thus they determine the time evolution dynamics. The final light direction $\mathbf{s}$ appears as an argument of the generalized spherical harmonics in Eqs. (35) and (37). Substituting Eqs. (18), (35), and (37) into Eqs. (28), (31), and then Eq. (24), the first and second cumulants in the SP representation are obtained as functions of $\mathbf{s}$ and t. The distribution function of polarized light is then expressed by Eq. (23), with Eq. (28) for the average center position and Eq. (24) for the width of the spread. Equation (23) produces the mth Stokes component of polarized light at position $\mathbf{r}$, with light direction $\mathbf{s}$, as a function of time t, initialed by $\mathbf{r}_0=0$, $\mathbf{s}_0=\hat{\mathbf{z}}$, and polarized state $\mathbf{I}^{\mathrm{SP}(0)}$ in an infinite uniform medium.

It is easy to reduce the above solution to the scalar (unpolarized) case by considering only the I_0 component. Because $\langle l,1,0,0|l,0\rangle=0$ in Eq. (34), Eqs. (35) and (37) can be greatly simplified. Also, Eq. (15) is reduced to $(2l+1)\exp(-\Pi^l_{00}t)/4\pi$ in the scalar case. Notice that the associated Legendre function $P^{(m)}_l(\mu)=(i)^m[(l+m)!/(l-m)!]^{1/2}P^l_{0,m}(\mu)$; our formula reduces to that given in Ref. [14] in the scalar case.

V. DISTRIBUTION FUNCTION ACCURATE UP TO AN ARBITRARY HIGH-ORDER CUMULANT

In order to calculate the polarized photon distribution function with accuracy up to an arbitrary high order, it is more convenient to set all spatial and angular vectors in the spherical harmonics basis, similar to Eq. (27), and to evaluate Eq. (22) via the CP:

$$\left[\left\langle\int_0^t dt_k\cdots\int_0^t dt_1\, T[s_{j_k}(t_k)\cdots s_{j_1}(t_1)]\right\rangle\right]^{\mathrm{SP}}_m =\frac{1}{F^{\mathrm{SP}}_m(\mathbf{s},\mathbf{s}_0,t)}[\mathbf{T}^{-1}\mathbf{G}(j_k,\ldots,j_1,t)\mathbf{T}\mathbf{I}^{\mathrm{SP}(0)}]_m, \quad (39)$$

with $j_1,\ldots,j_k=1,0,-1$ and $\mathbf{G}(j_k,\ldots,j_1,t)$ given by

$$\mathbf{G}(j_k,\ldots,j_1,t)=\left\{\int_0^t dt_k\int_0^{t_k}dt_{k-1}\cdots\int_0^{t_2}dt_1\int d\mathbf{s}^{(k)}\int d\mathbf{s}^{(k-1)}\cdots\int d\mathbf{s}^{(1)}\right.$$
$$\left.\times\mathbf{F}(\mathbf{s},\mathbf{s}^{(k)},t-t_k)s_{j_k}^{(k)}\mathbf{F}(\mathbf{s}^{(k)},\mathbf{s}^{(k-1)},t_k-t_{k-1})s_{j_{k-1}}^{(k-1)}\cdots\mathbf{F}(\mathbf{s}^{(2)},\mathbf{s}^{(1)},t_2-t_1)s_{j_1}^{(1)}\mathbf{F}(\mathbf{s}^{(1)},\mathbf{s}_0,t_1-0)+\text{perm.}\right\},\tag{40}$$

where $F_{mn}(\mathbf{s}^{(i)},\mathbf{s}^{(i-1)},t_i-t_{i-1})$ is given by Eq. (12). Using the GSF recurrence relation, Eq. (33), and the orthogonality relation of GSF's, Eq. (8), the integrals over $d\mathbf{s}^{(k)}\cdots d\mathbf{s}^{(1)}$ in Eq. (40) can be analytically performed. We obtain, when the initial $\mathbf{s}_0$ is along $\mathbf{z}$ and the final $\mathbf{s}=(\theta,\phi)$, that

$$[G(j_k,\ldots,j_1,t)]_{mn_0}=\left\{\sum_l P^l_{m,n_0-\Sigma_{f=1}^k j_f}(\cos\theta)\exp\left[-i\left(n_0-\sum_{f=1}^k j_f\right)\phi\right]\left[\prod_{f=1}^k\gamma_{j_f}\right]\sum_{n_k}\cdots\sum_{n_1}\sum_{h_k}\cdots\sum_{h_1}\frac{2(l-\Sigma_{f=1}^k h_f)+1}{4\pi}\right.$$
$$\times H^{l,h_k,\ldots,h_1}_{mn_k,\ldots,n_1n_0}(t)\prod_{g=1}^k\left\langle l-\sum_{f=1}^g h_{k-f+1},1,n_{k-g+1},0\middle| l-\sum_{f=1}^{g-1}h_{k-f+1},n_{k-g+1}\right\rangle$$
$$\left.\times\left\langle l-\sum_{f=1}^g h_{k-f+1},1,n_0-\sum_{f=1}^{k-g}j_f,-j_{k-g+1}\middle| l-\sum_{f=1}^{g-1}h_{k-f+1},n_0-\sum_{f=1}^{k-g+1}j_f\right\rangle\right\}+\text{perm},\tag{41}$$

with $n_f=2,0,-0,-2$ and $h_f=1,0,-1$, $f=1,2,\ldots,k$, with

$$H^{l,h_k,\ldots,h_1}_{mn_k,\ldots,n_1n_0}(t)=\exp(-\mu_a t)\sum_{i_{k+1}=1}^4\cdots\sum_{i_1=1}^4[B^l_{mn_k}]_{i_{k+1}}[B^{l-h_k}_{n_kn_{k-1}}]_{i_k}\cdots[B^{l-\Sigma_{f=1}^k h_{k-f+1}}_{n_1n_0}]_{i_1}$$
$$\times\int_0^t dt_k\int_0^{t_k}dt_{k-1}\cdots\int_0^{t_2}dt_1\exp[-\lambda^l_{i_{k+1}}(t-t_k)]\exp[-\lambda^{l-h_k}_{i_k}(t_k-t_{k-1})]\cdots\exp[-\lambda^{l-\Sigma_{f=1}^k h_{k-f+1}}_{i_1}(t_1-0)].\tag{42}$$

Note that all ensemble averages have been performed. Equation (42) involves integrals of exponential functions, which can be analytically performed. An explicit expression for evaluating integrals in Eq. (42) is presented in the Appendix B. Equation (42) involves all related scattering and absorption parameters and determines the time evolution dynamics. The final direction of light, $\mathbf{s}$, appears as an argument of GSF's in Eq. (41). Substituting Eq. (42) into Eq. (41), through Eq. (39), which transfers to the SP representation and introduces the initial polarized condition, and using a standard cumulant procedure, the cumulants as functions of angle $\mathbf{s}$ and time t up to an arbitrary kth order in the SP representation can be recursively obtained. The final position $\mathbf{r}$ appears in Eq. (20), and its components can be expressed on a spherical harmonics basis, similar to Eq. (27). Then, performing a numerical three-dimensional inverse Fourier transform over $\mathbf{q}$, an approximate distribution function $I_m^{\mathrm{SP}}(\mathbf{r},\mathbf{s},t)$ in the SP representation, accurate up to kth cumulant, can be calculated.

VI. DISCUSSION

In Sec. III, we derived an explicit expression of the polarized photon distribution function, which guarantees the exact average central position (the first cumulant) and the exact width of spread (the second cumulant). Moreover, with an increase of collision events or time, the distribution approaches accuracy in detail since the higher cumulants become relatively small compared to the appropriate power of the second cumulant. If we examine the spatial displacement after each collision event as an independent random variable $\Delta\mathbf{r}_i$, the total displacement is $\Sigma\Delta\mathbf{r}_i$ $(i=1,\ldots,N)$, with N the number collision events, which can be estimated by t/μ_s. If we define $\mathbf{Y}=(N)^{-1/2}\Sigma\Delta\mathbf{r}_i$, the central limit theorem claims that if N is a large number, then $\langle\mathbf{Y}^n\rangle_c/\langle\mathbf{Y}^2\rangle_c\sim N^{1-n/2}$, $n\geq 3$. Therefore, the sum of N variables will have an essentially Gaussian distribution. At early times, the photon's spread is narrow: hence, in many applications the detailed shape is less important than the correct position and correct narrow width of the beam, because of the finite resolution of detection devices. In case a more accurate distribution at early times is needed, Sec. IV provides formulas for analytically calculating the higher cumulants up to an arbitrary kth order. Then, performing a numerical three-dimensional Fourier transform, the distribution function accurate up to the kth order cumulant approximation can be obtained.

In summary, we present an analytical solution of the time-dependent polarized radiative transport equation in an infinite uniform isotropic medium. The Green's function for the angular part is exact. Using a cumulant expansion, we can analytically calculate the spatial cumulants up to an arbitrary high order. By terminating at the second order, we have derived an explicit expression of the polarized light distribution function. This expression is quantitatively accurate up to the second order cumulant approximation. Namely, the center position and the half-width are always exact and not modified when higher-order cumulants are added. The central limit theorem claims that after enough collision events, all cumulants higher than second approach small values, and the Gaussian spatial distribution calculated approaches accuracy in detail. Our results are given in terms of a distribution with coefficients that can be calculated algebraically, with moderate effort at the second cumulant level and additional effort to induce the third- and higher-order cumulants. This analytical solution provides a background distribution function for further study of optical tomography using polarized light.

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APPENDIX A

In this appendix we calculate $[B^l_{mn_0}]_i$ in Eq. (15). Substituting Eq. (15) into Eq. (14), we obtain a set of linear homogeneous equations

$$\sum_n [\Pi^l_{mn} - \lambda^l_i \delta_{m,n}][B^l_{nn_0}]_i = 0, \tag{A1}$$

where eigenvalues λ^l_i $(i=1,2,3,4)$ are given by Eq. (16). These equations, however, are not linearly independent. Adding the initial condition, Eq. (14b), given by

$$\sum_i [B^l_{mn_0}]_i = \delta_{m,n_0}, \tag{A2}$$

the unique solution of $[B^l_{mn_0}]_i$ then can be obtained. For given n_0 and l, 16 components of $[B^l_{mn_0}]_i$ construct a column vector in the space of the direct product of $i \times m$. Combining Eqs. (A1) and (A2) in $i \times m$ space, we obtain the following matrix equation:

$$\mathbf{AB} = \mathbf{C}. \tag{A3}$$

A is a 16×16 matrix:

$$A_{i\times m, j\times n} = [\Pi^l_{mn}\delta_{i,j} - \lambda^l_i \delta_{i,j}\delta_{m,n} + \delta_{m,n}]. \tag{A4}$$

B and **C** are 16×1 column vectors: $B_{j\times n} = [B^l_{nn_0}]_j$ and $C_{i\times m} = \delta_{m,n_0}$. Here **A** and **C** are given, while **B** is unknown. Equation (A3) is a standard form of a group of 16 linear equations. The solution is given by

$$B_{i\times m} = \Delta_{i\times m}/\det(\mathbf{A}), \tag{A5}$$

with $\Delta_{i\times m}$ is obtained by replacing the $(i\times m)$th column in the determinant of **A** by the column vector **C**.

APPENDIX B

In this appendix, we derive an analytical expression for Eq. (42) to kth order. By defining

$$b_g \equiv \lambda_{i_{g+1}}^{[l-\Sigma_{f=1}^{k-g} h_{k-f+1}]} - \lambda_{i_g}^{[l-\Sigma_{f=1}^{k-g+1} h_{k-f+1}]}, \quad g=1,\ldots,k, \tag{B1}$$

Eq. (42) can be written as

$$H^{l,h_k,\ldots,h_1}_{mn_k,\ldots,n_1n_0}(t) = \exp(-\mu_a t)\sum_{i_{k+1}=1}^{4}\cdots\sum_{i_1=1}^{4}[B^l_{mn_k}]_{i_{k+1}}[B^{l-h_k}_{n_kn_{k-1}}]_{i_k}\cdots[B^{l-\Sigma_{f=1}^{k}h_{k-f+1}}_{n_1n_0}]_{i_1}\exp(-\lambda^l_{i_{k+1}}t)F^{(k)}(t), \tag{B2}$$

with

$$F^{(k)}(t) = \int_0^t dt_k\, e^{b_k t_k}\int_0^{t_k} dt_{k-1}\, e^{b_{k-1}t_{k-1}}\cdots\int_0^{t_2} dt_1\, e^{b_1 t_1}. \tag{B3}$$

It is easy to directly calculate Eq. (B3) for a few low k orders:

$$F^{(1)}(t) = \frac{e^{b_1 t}}{b_1} - \frac{1}{b_1}, \tag{B4a}$$

$$F^{(2)}(t) = \frac{e^{(b_1+b_2)t}}{b_1(b_1+b_2)} - \frac{e^{b_2 t}}{b_1 b_2} + \frac{1}{(b_1+b_2)b_2}, \tag{B4b}$$

$$F^{(3)}(t) = \frac{e^{(b_1+b_2+b_3)t}}{b_1(b_1+b_2)(b_1+b_2+b_3)} - \frac{e^{(b_2+b_3)t}}{b_1b_2(b_2+b_3)} + \frac{e^{b_3 t}}{(b_1+b_2)b_2b_3} - \frac{1}{(b_1+b_2+b_3)(b_2+b_3)b_3}. \tag{B4c}$$

In each step of integration, the difficulty is in determining the constant term. In the following we prove that this term is given by $(-1)^k/[b_k(b_k+b_{k-1})\cdots(b_k+b_{k-1}+\cdots+b_1)]$. Equation (B3) can be written as

$$F^{(k)}(t)=\int_0^t dt'\, e^{b_k t'} F^{k-1}(t'). \tag{B5}$$

Using integration by parts to Eq. (B5), we obtain

$$F^{(k)}(t)=\frac{1}{b_k}\left[e^{b_k t}F^{(k-1)}(t)-\int_0^t dt'\, e^{(b_k+b_{k-1})t'}F^{(k-2)}(t')\right]. \tag{B6}$$

Recursively applying Eq. (B6), we obtain

$$F^{(k)}(t)=\frac{e^{b_k t}}{b_k}F^{(k-1)}(t)-\frac{e^{(b_k+b_{k-1})t}}{b_k(b_k+b_{k-1})}F^{(k-2)}(t)$$
$$+\cdots+(-1)^f\frac{e^{(b_k+b_{k-1}+\cdots+b_{k-f})t}}{b_k(b_k+b_{k-1})\cdots(b_k+b_{k-1}+\cdots+b_{k-f})}F^{k-f-1}(t)$$
$$+\cdots+(-1)^{k-1}\frac{e^{(b_k+b_{k-1}+\cdots+b_1)t}-1}{b_k(b_k+b_{k-1})\cdots(b_k+b_{k-1}+\cdots+b_1)}. \tag{B7}$$

Equation (B7) provides a formula to recursively evaluate Eq. (40) up to kth order. Also, Eq. (B7) produces the above-mentioned constant term. An explicit expression of Eq. (42) can then be written as

$$H^{1,h_k,\ldots,h_1}_{mn_k,\ldots,n_1n_0}(t)=\exp(-\mu_a t)\sum_{i_{k+1}=1}^{4}\cdots\sum_{i_1=1}^{4}[B^l_{mn_k}]_{i_{k+1}}[B^{l-h_k}_{n_kn_{k-1}}]_{i_k}\cdots[B^{l-\Sigma_{g=1}^k h_{k-g+1}}_{n_1n_0}]_{i_1}$$
$$\times\exp(-\lambda^l_{i_{k+1}}t)\sum_{g=0}^{k}\frac{(-1)^g\exp[\Sigma_{f=0}^{k-g}b_{k-f+1}t]}{\Pi_{j=1}^k L_j^{(g)}}, \tag{B8}$$

with $b_{k+1}\equiv 0$, and

$$L_j^{(g)}=\sum_{f=j}^{g}b_f,\quad j\leqslant g,\quad \text{or}\quad L_j^{(g)}=\sum_{f=g+1}^{j}b_f,\quad j>g, \tag{B9}$$

where b_g is defined in Eq. (B1).

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