Photon-transport forward model for imaging in turbid media

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A photon-transport forward model for image reconstruction in turbid media is derived that treats weak inhomogeneities through a Born approximation of the Boltzmann radiative transfer equation. This model can conveniently replace the commonly used diffusion approximation in optical tomography. An analytical expression of the background Green's function is obtained from the cumulant solution of the Boltzmann equation. Our model provides the correct behavior of photon migration at early times and reduces at long times to the center-moved diffusion approximation. Numerical comparisons between this model and the standard and center-moved diffusion models are presented.

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Probing the internal properties of highly scattering turbid media with photons has a variety of applications in geophysics, radio astronomy, and medical tomography. Imaging based on the diffusion approximation has been pursued over the past decade because it captures the core characteristic of light migration in turbid media and is easy to implement.[1–3] However, in the diffusion approximation, light is assumed to diffuse from a fixed source with a constant diffusion coefficient throughout the full time range when photons propagate inside the uniform medium.[4] This assumption is invalid when the incident photon retains its early time directionality preference. To account for this difficulty, a common practice is to assume that all the incident photons are initially scattered at a depth $z_0 = l_t$ (transport mean free path) inside the turbid medium,[5] which we call the center-moved diffusion model (CDM). But CDM breaks the reciprocity theorem and still fails in the description of photon propagation at early times.[6]

To fully account for photon migration in a turbid medium, one must use the radiative transfer equation instead:

$$\frac{\partial}{\partial t} I(\mathbf{r},\mathbf{s},t) + c\mathbf{s}\cdot\nabla_{\mathbf{r}} I(\mathbf{r},\mathbf{s},t) + c[\mu_s(\mathbf{r}) + \mu_a(\mathbf{r})]I(\mathbf{r},\mathbf{s},t) = c\mu_s(\mathbf{r})\int \mathrm{d}\mathbf{s}' P(\mathbf{s},\mathbf{s}')I(\mathbf{r},\mathbf{s}',t)\mathrm{d}\mathbf{s}' + q(\mathbf{r},\mathbf{s},t), \quad (1)$$

where $I(\mathbf{r}, \mathbf{s}, t)$ is the photon-distribution function depending on position $\mathbf{r}$, direction $\mathbf{s}$, and time t; c is light speed inside the medium; μ_a and μ_s denote the position-dependent absorption and scattering coefficients; $q(\mathbf{r}, \mathbf{s}, t)$ is the photon-source strength; and $P(\mathbf{s}, \mathbf{s}')$ is the normalized phase function of light propagation in the medium.

Recently, we derived an analytical solution for the photon distribution, $I^{(0)}(\mathbf{r}, \mathbf{s}, t)$, and photon density, $N^{(0)}(\mathbf{r}, t)$, in an infinite uniform medium, with exact spatial center position and exact spatial half-width, at any direction and time[7]; the exact solution up to an arbitrary order of cumulants was also derived.[8] The photon-density distribution is found to have a center that advances in time and an ellipsoidal contour that grows and changes shape, providing a clear picture of the time evolution of light migration from the initial ballistic to the final diffusive regime. A forward model adding inhomogeneity to the analytical expression of $I^{(0)}(\mathbf{r}, \mathbf{s}, t)$ involves a complicated numerical integration over angular parameters. In this Letter we use an approximate expression of $I^{(0)}(\mathbf{r}, \mathbf{s}, t)$ that not only retains the main features of photon propagation in both early and long time limits but also is convenient for building a forward model to account for weak inhomogeneities of the medium that are treatable in the Born approximation.

The photon distribution in an infinite uniform medium, $I^{(0)}(\mathbf{r}, \mathbf{s}, t)$, is assumed to have the following form:

$$I^{(0)}(\mathbf{r},\mathbf{s},t|\mathbf{r}_0,\mathbf{s}_0,t_0) = N^{(0)}(\mathbf{r},t|\mathbf{r}_0,\mathbf{s}_0,t_0)F^{(0)}(\mathbf{s},t|\mathbf{s}_0,t_0) - \frac{3}{4\pi}D(t-t_0)\mathbf{s}\cdot\nabla_{\mathbf{r}}N^{(0)}(\mathbf{r},t|\mathbf{r}_0,\mathbf{s}_0,t_0), \quad (2)$$

where $N^{(0)}(\mathbf{r}, t|\mathbf{r}_0, \mathbf{s}_0, t_0)$ is the photon density for a point pulse propagating along $\mathbf{s}_0$ at position $\mathbf{r}_0$ and time t_0 in an infinite uniform medium, $F^{(0)}(\mathbf{s}, t|\mathbf{s}_0, t_0)$ is the known exact photon distribution in light-direction space, and $D(t - t_0)$ is the time-dependent diffusion coefficient. The full definitions of these quantities originated in Ref. 7 and are given as follows: $F^{(0)}(\mathbf{s}, t|\mathbf{s}_0, t_0) = (4\pi)^{-1}\sum_l (2l + 1)\exp[-g_l(t - t_0)]P_l(\mathbf{s}\cdot\mathbf{s}_0)$, where $g_l = c\mu_s[1 - a_l/(2l + 1)]$, a_l are the coefficients in the Legendre expansion of the phase function, $P(\mathbf{s}, \mathbf{s}') = (4\pi)^{-1}\sum_l a_l P_l(\mathbf{s}\cdot\mathbf{s}')$, especially, $g_0 = 0$ and $g_1 = c\mu_s{}'$, where $\mu_s{}'$ is the reduced scattering coefficient. The diffusion coefficient $D(t)$ is taken to be an average of the time-dependent semimajor axis, D_{zz}, and semiminor axes $D_{xx} = D_{yy}$ of the diffusion coefficient ellipsoid:

$$D(t) = \frac{D_{xx} + D_{yy} + D_{zz}}{3} = \frac{c}{3t}\left\{\frac{t}{g_1} - \frac{1 - \exp(-g_1 t)}{g_1{}^2} - \frac{[1 - \exp(-g_1 t)]^2}{2g_1{}^2}\right\}. \quad (3)$$

The photon density is given by

$$N^{(0)}(\mathbf{r}, t|\mathbf{r}_0, \mathbf{s}_0, t_0) = \frac{1}{[4\pi D(t - t_0)c(t - t_0)]^{3/2}} \times \exp\left\{-\frac{[\mathbf{r} - \mathbf{r}_0 - \mathbf{s}_0\Delta(t - t_0)]^2}{4D(t - t_0)c(t - t_0)}\right\}\exp[-\mu_a c(t - t_0)], \tag{4}$$

where $\Delta(t) = c[1 - \exp(-g_1 t)]/g_1$ is the average center of photons, which moves with speed c initially and stops at $c/g_1 = l_t$ in the long time limit.

At the early times $t \to t_0$, the first term of Eq. (2) dominates, and $F^{(0)}(\mathbf{s}, t|\mathbf{s}_0, t_0) \to \delta(\mathbf{s} - \mathbf{s}_0)$, $D(t - t_0) \to c^2(t - t_0)^2\mu_s'/9 \to 0$, and $N^{(0)}(\mathbf{r}, \mathbf{t}|\mathbf{r}_0, \mathbf{s}_0, t_0) \to \delta[\mathbf{r} - \mathbf{r}_0 - c(t - t_0)\mathbf{s}_0]$; thus $I^{(0)}(\mathbf{r}, \mathbf{s}, t|\mathbf{r}_0, \mathbf{s}_0, t_0)$ provides a correct picture of ballistic motion at the speed of light along the incident direction, $\mathbf{s}_0$. In the long time limit, $F^{(0)}(\mathbf{s}, t|\mathbf{s}_0, t_0) \to (4\pi)^{-1}$, $D(t) \to (3\mu_s')^{-1}$, Eq. (2) reduces to the photon distribution of the CDM approximation.

A perturbative method is then used to obtain the forward model when weak inhomogeneities are introduced into the otherwise uniform medium. Making a perturbation expansion of the radiative transfer equation (1) to the first-order Born approximation, we derive the change in photon distribution from $I^{(0)}(\mathbf{r}, \mathbf{s}, t)$, which is the photon distribution of the uniform background, as

$$\begin{aligned}\delta I(\mathbf{r}, \mathbf{s}, t|\mathbf{r}_0, \mathbf{s}_0, t_0) &= \int \mathrm{d}t' \int \mathrm{d}\mathbf{r}' \int \mathrm{d}\mathbf{s}' I^{(0)}(\mathbf{r}', -\mathbf{s}', t - t'|\mathbf{r}, -\mathbf{s}) \\ &\times \left\{\int \mathrm{d}\mathbf{s}'' c\,\delta[\mu_s P](\mathbf{s}', \mathbf{s}'', \mathbf{r}') I^{(0)}(\mathbf{r}', \mathbf{s}'', t - t_0|\mathbf{r}_0, \mathbf{s}_0) \right. \\ &\left. - c[\delta\mu_s(\mathbf{r}') + \delta\mu_a(\mathbf{r}')] I^{(0)}(\mathbf{r}', \mathbf{s}', t' - t_0|\mathbf{r}_0, \mathbf{s}_0)\right\},\end{aligned} \tag{5}$$

where $\delta\mu_a$, $\delta\mu_s$, and $\delta[\mu_s P]$ are the changes of the absorption coefficient, the scattering coefficient, and the angular-dependent differential scattering coefficient, respectively, from the background to the inhomogeneity. The optical reciprocity theorem, $I^{(0)}(\mathbf{r}, \mathbf{s}, t - t'|\mathbf{r}', \mathbf{s}') = I^{(0)}(\mathbf{r}', -\mathbf{s}', t - t'|\mathbf{r}, -\mathbf{s})$, is used to obtain Eq. (5).

Expanding $\delta[\mu_s P]$ in Legendre polynomials and substituting Eq. (2) into Eq. (5), the integrations over angular variables in Eq. (5) can be analytically performed. We obtain

$$\begin{aligned}\delta I(\mathbf{r}, \mathbf{s}, t|\mathbf{r}_0, \mathbf{s}_0, t_0) &= -\frac{1}{4\pi} \int \mathrm{d}t' \int \mathrm{d}\mathbf{r}' c\,\delta\mu_a(\mathbf{r}') \\ &\times N^{(0)}(\mathbf{r}', t - t'|\mathbf{r}, -\mathbf{s}) N^{(0)}(\mathbf{r}', t - t_0|\mathbf{r}_0, \mathbf{s}_0) \\ &+ \frac{3c}{4\pi} \int \mathrm{d}t' \int \mathrm{d}\mathbf{r}' D(t - t') D(t' - t_0)\,[\delta\mu_a(\mathbf{r}') + \delta\mu_s'(\mathbf{r}')] \\ &\times \nabla_{\mathbf{r}'} N^{(0)}(\mathbf{r}', t - t'|\mathbf{r}, -\mathbf{s}) \cdot \nabla_{\mathbf{r}'} N^{(0)}(\mathbf{r}', t' - t_0|\mathbf{r}_0, \mathbf{s}_0) \\ &+ \frac{3c}{4\pi} \int \mathrm{d}t' \int \mathrm{d}\mathbf{r}' D(t - t')\,[\delta\mu_a(\mathbf{r}') + \delta\mu_s'(\mathbf{r}')] \exp(-g_1 t') \\ &\times \{N^{(0)}(\mathbf{r}', t - t_0|\mathbf{r}, -\mathbf{s})\mathbf{s} \cdot \nabla_{\mathbf{r}'} N^{(0)}(\mathbf{r}', t - t'|\mathbf{r}_0, \mathbf{s}_0) \\ &- \mathbf{s}_0 \cdot \nabla_{\mathbf{r}'} N^{(0)}(\mathbf{r}', t - t'|\mathbf{r}, -\mathbf{s}) N^{(0)}(\mathbf{r}', t' - t_0|\mathbf{r}_0, \mathbf{s}_0)\}\end{aligned} \tag{6}$$

after neglecting fast-decaying terms involving $\exp(-2g_l t)$ for $l \geq 1$.

The photon-transport forward model (PTFM), Eq. (6), is the main result in this Letter and obeys the reciprocal relation $\delta I(\mathbf{r}, \mathbf{s}, t - t_0|\mathbf{r}_0, \mathbf{s}_0) = \delta I(\mathbf{r}_0, -\mathbf{s}_0, t - t_0|\mathbf{r}, -\mathbf{s})$. In the long time limit, the term in Eq. (6) that contains the exponential decay factor $\exp(-g_1 t')$ can be neglected, and the change in photon density, $4\pi\delta I(\mathbf{r}, \mathbf{s}, t)$ in the PTFM, is reduced to the diffuse imaging model [Eq. (14) in Ref. 9].

To show the difference between our model and the diffusion models, we consider a point photon pulse $\delta(\mathbf{r})\delta(\mathbf{s} - \hat{z})\delta(t)$ propagating inside an infinite scattering turbid medium with absorption coefficient $\mu_a = 0$, reduced scattering coefficient $\mu_s' = 1\ \text{mm}^{-1}$, anisotropy $g = 0.9$, and refractive index $n = 1.33$. These optical properties are similar to those of a typical breast tissue. A Henyey–Greenstein phase function is adopted in the following calculations.[10]

The time-resolved profiles of transmission $I^{(0)}(\mathbf{r}, \hat{z}, t)$ at position $\mathbf{r} = (0, 0, 5)$ mm and backscattering $I^{(0)}(\mathbf{r}, -\hat{z}, t)$ at position $\mathbf{r} = (0, 2, 0)$ mm are shown by the solid curves in Figs. 1(a) and 1(b), respectively. For comparison, the photon density divided by 4π from the original diffusion model (ODM) and the CDM are also plotted.

In a transmission case [Fig. 1(a)], the peak photon intensity in the ODM arrives too late compared with the experimental result reported by Yoo *et al.*,[11] whereas in the CDM the artificial adjustment of the source position for l_t leads to an arrival of photons faster than light speed. The intensity at forward directions from our model is stronger than that of diffusion models, which indicates that a ceratin anisotropic angular distribution remains even at a distance of $5l_t$ from the source.

In the case of backscattering [Fig. 1(b)], photons diffuse from the origin (0, 0, 0) to (0, 2, 0) mm with a constant diffusion coefficient $D = l_t/3$ in the ODM, whereas photons diffuse from the adjusted source position (0, 0, 1) mm to (0, 2, 0) mm with the same constant diffusion coefficient in the CDM. The photons in our model are backscattered to (0, 2, 0) mm later than those in the ODM and the CDM because the center of photons moves forward along the positive z direction and diffuses from the moving center with a gradually increasing diffusion coefficient from 0 to $l_t/3$ in the PTFM.

Consider a forward model with a scattering inhomogeneity $\delta\mu_s' = 0.1\ \text{mm}^{-1}$ of unit volume placed at position (0, 0, 2) mm. The time-resolved profiles of $-\delta I(\mathbf{r}, \hat{z}, t)$ at position $\mathbf{r} = (0, 0, 5)$ mm and $\delta I(\mathbf{r}, -\hat{z}, t)$ at position $\mathbf{r} = (0, 2, 0)$ mm from Eq. (6) and those from diffusion models are shown in Fig. 2. The significant difference between our model and diffusion models shows that the nondiffusive nature of photon migration at early times is important and cannot be neglected when the separation of any pair of source, inhomogeneity, and detector is small.

In conclusion, a photon-transport forward model of image reconstruction in turbid media has been derived by use of the Born approximation of th radiative transfer equation. A simplified cumulant solution in an

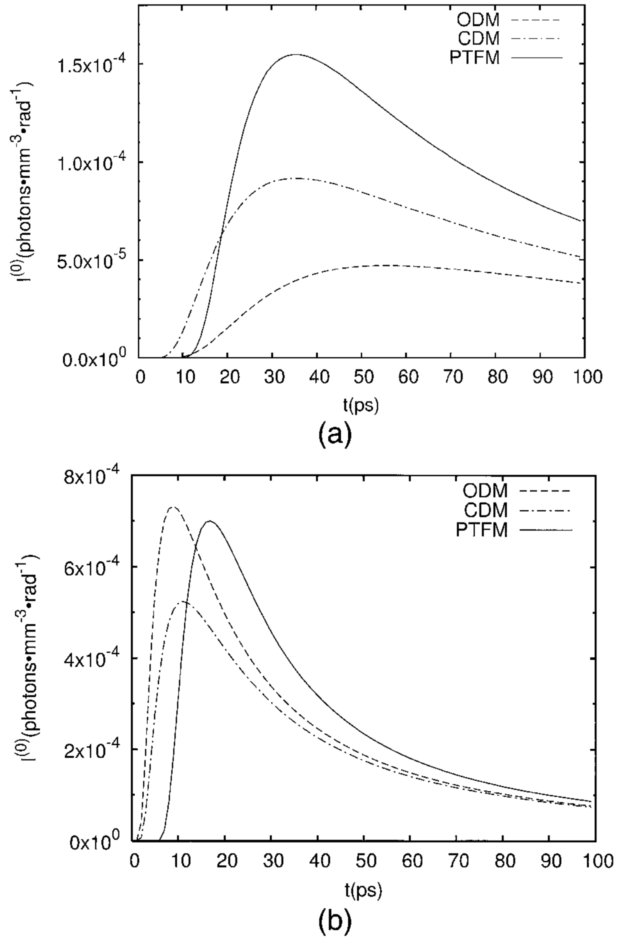

Fig. 1. Photon-distribution profiles (a) $I^{(0)}(\mathbf{r}, \hat{z}, t)$ at position $\mathbf{r} = (0, 0, 5)$ mm and (b) $I^{(0)}(\mathbf{r}, -\hat{z}, t)$ at position $\mathbf{r} = (0, 2, 0)$ mm.

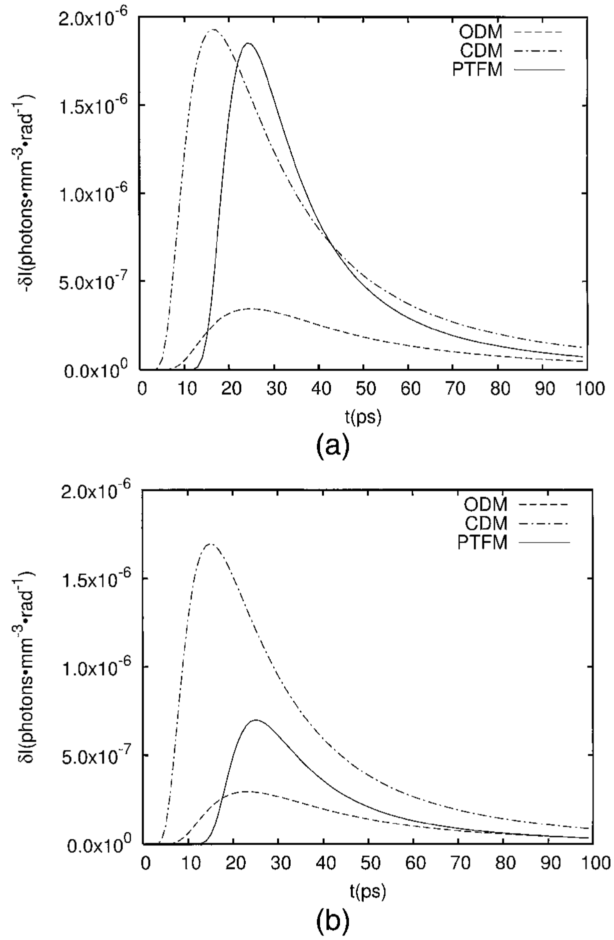

Fig. 2. Changes of photon-distribution function (a) $-\delta I(\mathbf{r}, \hat{z}, t)$ at position $\mathbf{r} = (0, 0, 5)$ mm and (b) $\delta I(\mathbf{r}, -\hat{z}, t)$ at position $\mathbf{r} = (0, 2, 0)$ mm.

infinite uniform medium serves as the background Green's function. This model provides a correct picture of nearly ballistic motion of photons at early times and reduces to the center-moved diffusion approximation at long times. Extension to semi-infinite and slab geometries of this model is being studied.

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