Spectral Extent and Pulse Shape of the Supercontinuum for Ultrashort Laser Pulse

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Abstract—**The pulse shape and the generated supercontinuum spectral distribution associated with the propagation of an ultrashort intense pulse in a cubic ($\chi^{(3)}$) nonlinear medium are obtained by solving the electromagnetic field wave equation using the method of multiple scales. New forms for the set of quasi-linear partial differential equations describing the system, pulse distortion, and the spectral extent of the supercontinuum are the focus of this paper.**

I. Introduction

Supercontinuum generation is the production of nearly white continuous spectrum by propagating picosecond and subpicosecond laser pulses through nonlinear media. Alfano and Shapiro [1] were the first to experimentally observe this phenomenon more than 15 years ago. They and others [2] used this superbroadening as a means to produce ultrashort pulses in the spectral range from ultraviolet to infrared. The shape, fine structure, and extent of the spectrum produced are functions of the nonlinear index of refraction of the medium, the shape, wavelength, duration, intensity, and phase modulation of the pump laser pulse, and the interaction length of the pulse in the medium. Typically, the observed broadened spectrum consists of larger frequency extent towards the blue than the red by factors of approximately two, a feature commonly referred to as spectral asymmetry. This coherent and ultrafast superbroad frequency band has been mainly used as a spectral tool for time-resolved absorption spectroscopy [3], [4] and nonlinear optical effects [3], [5]. Recently, new uses in engineering applications [6], such as ranging, 3-D imaging, atmospheric remote sensing, and optical fiber characterization, have been proposed.

The supercontinuum was explained either as a result of self-phase modulation (SPM) following self-focusing and optical breakdown [7] or as a result of a four-wave parametric process [8]. The asymmetry in the Stokes and anti-Stokes regions was attributed to contributions from plasmas and/or the time response of the nonlinear index of refraction. Recent experiments [9] suggest, however, that the supercontinuum can be observed in experimental conditions where self-focusing and optical breakdown are absent, and where the measured spectrum is claimed not to exhibit characteristics peculiar to the resonant four-wave parametric process. Similarly, in recent experiments [10]–[12] on ultrashort pulse propagation in long optical fibers, it was found that the asymmetry in the outgoing pulse shape was due to the distortion induced by propagation in the medium rather than due to asymmetry in the input pulse.

A more rigorous theoretical analysis of the wave equation with third-order nonlinearity in the case of negligible dispersions is needed. The traditional self-phase modulation (SPM) theory, which is essentially a first-order slowly varying amplitude (SVA) perturbation theory, predicts spectral extents smaller than those experimentally observed and does not differentiate between the Stokes and anti-Stokes sides. Likewise, traditional SPM theory does not predict the asymmetry observed in the shape of pulses propagating in long optical fibers. In attempts to improve on the traditional theory, Anderson and Lisak [13], in the framework of the SVA approximation, obtained an analytic expression for the pulse distortion. Yang and Shen [14] derived a quasi-linearized form for the partial differential equations, valid for more general conditions than the SVA approximation, and obtained solutions that exhibit spectral distribution asymmetry. Our group [15] independently obtained, through a preliminary use of the method of multiple scales and under the assumption of factorizability of the amplitude, an approximate expression for the frequency extents with the desired asymmetry. In this paper, we improve on the previous treatments—assumptions on the relative magnitude of the derivative of the amplitude versus the phase derivative are not made—the quasi-linearized forms of the partial differential equations are consistently derived in a systematic expansion and expressions for the pulse distortion and the spectral extent of the supercontinuum, that can be directly compared to measurable quantities, are derived.

The paper is organized as follows. Section II reviews the physical problem, the nonlinear wave equation, and the pertinent physical parameters. Section III summarizes the method of multiple scales. In Section IV, the quasi-linearized partial differential equations corresponding to the nonlinear wave equations are derived and their forms are compared to previously used equations. The pulse ampli-

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TABLE I
RANGE OF EXPERIMENTAL PARAMETERS IN SUPERCONTINUUM GENERATION EXPERIMENTS (SOURCE WAVELENGTH = 1 μm, PULSE WIDTH = 10^{-13} s, AND BEAM DIAMETER AT INPUT PLANE = 1 mm)

P_c Critical power for self-focusing	$n_2 = 10^{-22}$ (MKS) $= 0.9 \times 10^{-13}$ (esu)			$n_2 = 10^{-20}$ (MKS) $= 0.9 \times 10^{-11}$ (esu)		
	2×10^6 W			2×10^4 W		
P_{in} Pulse input power	10^7 W	10^9 W	10^{11} W	10^7 W	10^9 W	10^{11} W
E_0 Maximum amplitude of the incoming pulse	9×10^7 V/m	9×10^8 V/m	9×10^9 V/m	9×10^7 V/m	9×10^8 V/m	9×10^9 V/m
S_f Distance to focusing point	0.4 m	0.04 m	0.004 m	0.04 m	0.004 m	0.0004 m
$\epsilon = \frac{n_2 \lvert E_0\rvert^2}{n}$ (n = 1.5)	$\frac{2}{3} \times 10^{-6}$	$\frac{2}{3} \times 10^{-4}$	$\frac{2}{3} \times 10^{-2}$	$\frac{2}{3} \times 10^{-4}$	$\frac{2}{3} \times 10^{-2}$	$\frac{2}{3}$
$\epsilon V = \frac{n_2 \lvert E_0\rvert^2 z}{c\tau}$ $z = 1$ μm of sample thickness	$\frac{1}{3} \times 10^{-7}$	$\frac{1}{3} \times 10^{-5}$	$\frac{1}{3} \times 10^{-3}$	$\frac{1}{3} \times 10^{-5}$	$\frac{1}{3} \times 10^{-3}$	$\frac{1}{3} \times 10^{-1}$

tude distortion is obtained in Section V. The frequency extents of the supercontinuum as a function of the incoming pulse characteristics is computed and compared in Section VI to other results.

II. NONLINEAR WAVE EQUATION

The Maxwell's equation in a nonlinear medium where the dispersion of the index of refraction and its imaginary part are neglected is given by

$$\nabla^2 \vec{E} - \frac{n^2}{c^2} \frac{\partial \vec{E}}{\partial t^2} = \frac{2nn_2}{c^2} \frac{\partial^2}{\partial t^2} \langle \vec{E} \cdot \vec{E} \rangle \vec{E} \tag{1}$$

where n is the linear index of refraction and n_2 is the nonlinear index of refraction. The wave equation reduces to

$$\frac{\partial^2 E}{\partial z^2} - \frac{1}{v_g^2} \frac{\partial^2 E}{\partial t^2} = \frac{nn_2}{c^2} \frac{\partial^2}{\partial t^2} |E|^2 E \tag{2}$$

under the assumption that one component of $\vec{E}$ is present and the transverse variation of $\vec{E}$ is neglected and $\langle \vec{E} \cdot \vec{E} \rangle = |E|^2/2$.

To reduce the differential equation to a dimensionless form, we introduce the new dimensionless variables Φ, T, Z defined as

$$T = t/\tau_L, \; Z = z/v_g\tau_L, \; E = E_0\Phi \tag{3}$$

where τ_L is the characteristic time associated with the pulse (pulse width), v_g is the group velocity in the medium, and E_0 is the maximum value of the electric field amplitude at the entrance plane of the medium. Introducing the dimensionless nonlinear coupling constant ϵ, defined as

$$\epsilon \equiv \frac{n_2 |E_0|^2}{n}, \tag{4}$$

the wave equation in dimensionless form reduces to

$$\left(\frac{\partial^2}{\partial Z^2} - \frac{\partial^2}{\partial T^2}\right) \Phi = \epsilon \frac{\partial^2}{\partial T^2} |\Phi|^2 \Phi. \tag{5}$$

Typical values for the above parameters are

$$\tau_L \sim 10^{-13} \text{ s}$$

$$n_2 \sim 10^{-22}\text{–}10^{-20} \text{ MKS } [10^{-13}\text{–}10^{-11} \text{ esu}]$$

$$v_g \sim 2 \times 10^8 \text{ m/s}.$$

The critical power for self-focusing [17] is

$$P_c \approx \frac{\pi\epsilon_0 c^3}{n_2\omega^2} \tag{6}$$

and the distance from the input plane to the self-focusing point is (for $P \gg P_c$)

$$S_f \sim \frac{ka^2}{2} \sqrt{\frac{P_c}{P}}. \tag{7}$$

Consequently, only if the sample thickness is smaller than S_f can one assert that the supercontinuum observed does not have its origin in self-focusing or optical breakdown. In Table I, we summarize the values of the different experimentally relevant parameters for an incoming pulse with the following characteristics: $\lambda = 10^{-6}$ m, $\tau_L \sim 10^{-13}$ s, and spot size $a \sim 10^{-3}$ m.

III. METHOD OF MULTIPLE SCALES

In this section, we summarize the method of multiple scales.

The functional dependence of Φ on Z, T, and ϵ in the solution of (5) is not disjoint. To first order in ϵ, Φ depends on the combinations ϵT and ϵZ as well as on the individual T, Z, and ϵ. Carrying the perturbation to higher orders, Φ

depends additionally on $\epsilon^2 T$, $\epsilon^2 Z$, $\epsilon^3 T$, $\epsilon^3 Z$, etc. Hence, it is convenient to write $\Phi(Z, T; \epsilon)$ as

$$\Phi(Z, T; \epsilon) = \Phi(Z_0, T_0, Z_1, T_1, Z_2, T_2, \cdots, \epsilon) \quad (8)$$

where the new scaled variables Z_1, T_1, Z_2, T_2, etc., are defined as

$$T_0 = T, \quad T_1 = \epsilon T, \quad T_2 = \epsilon^2 T \cdots$$
$$Z_0 = Z, \quad Z_1 = \epsilon Z, \quad Z_2 = \epsilon^2 Z \cdots.$$

T_n's and Z_n's represent different time and distance scales. Φ is determined as a function of the old and new variables. Next, we seek a uniform expansion solution to Φ in the form

$$\begin{aligned}\Phi = &\Phi_0 (T_0, Z_0, T_1, Z_1, T_2, Z_2 \cdots)\\ &+ \epsilon\, \Phi_1(T_0, Z_0, T_1, Z_1, T_2, Z_2 \cdots)\\ &+ \epsilon^2\, \Phi_2(T_0, Z_0, T_1, Z_1, T_2, Z_2 \cdots).\end{aligned} \quad (9)$$

To express the derivatives in (5) as function of the new variables, we use the chain rules for derivatives; then

$$\begin{aligned}\frac{\partial}{\partial T} &= \frac{\partial}{\partial T_0} + \epsilon \frac{\partial}{\partial T_1} + \epsilon^2 \frac{\partial}{\partial T_2} + \cdots\\ \frac{\partial}{\partial Z} &= \frac{\partial}{\partial Z_0} + \epsilon \frac{\partial}{\partial Z_1} + \epsilon^2 \frac{\partial}{\partial Z_2} + \cdots\\ \frac{\partial^2}{\partial T^2} &= \frac{\partial^2}{\partial T_0^2} + 2\epsilon \frac{\partial^2}{\partial T_0 \partial T_1} + \epsilon^2\left(2 \frac{\partial^2}{\partial T_0 \partial T_2} + \frac{\partial^2}{\partial T_1^2}\right) + \cdots\\ \frac{\partial^2}{\partial Z^2} &= \frac{\partial^2}{\partial Z_0^2} + 2\epsilon \frac{\partial^2}{\partial Z_0 \partial Z_1} + \epsilon^2\left(2 \frac{\partial^2}{\partial Z_0 \partial Z_2} + \frac{\partial^2}{\partial Z_1^2}\right) + \cdots.\end{aligned} \quad (10)$$

Using the expressions (9) and (10) in the partial differential equation (5), and equating the respective coefficients of ϵ^n, one obtains, respectively, for the terms multiplying ϵ^0, ϵ, and ϵ^2 the following equations:

$$\left(\frac{\partial^2}{\partial Z_0^2} - \frac{\partial^2}{\partial T_0^2}\right) \Phi_0 = 0, \quad (11)$$

$$\left(\frac{\partial^2}{\partial Z_0^2} - \frac{\partial^2}{\partial T_0^2}\right) \Phi_1 + 2\left(\frac{\partial^2}{\partial Z_1 \partial Z_0} - \frac{\partial^2}{\partial T_1 \partial T_0}\right) \Phi_0 = \frac{\partial^2}{\partial T_0^2} |\Phi_0|^2 \Phi_0, \quad (12)$$

and

$$\begin{aligned}&\left(\frac{\partial^2}{\partial Z_0^2} - \frac{\partial^2}{\partial T_0^2}\right) \Phi_2 + 2\left(\frac{\partial^2}{\partial Z_1 \partial Z_0} - \frac{\partial^2}{\partial T_1 \partial T_0}\right) \Phi_1\\ &+ \left(\frac{\partial^2}{\partial Z_1^2} - \frac{\partial^2}{\partial T_1^2} + 2 \frac{\partial^2}{\partial Z_0 \partial Z_2} - 2 \frac{\partial^2}{\partial T_0 \partial T_2}\right) \Phi_0\\ &= 2 \frac{\partial^2}{\partial T_1 \partial T_0} |\Phi_0|^2 \Phi_0 + \frac{\partial^2}{\partial T_0^2} |\Phi_1|^2 \Phi_1.\end{aligned} \quad (13)$$

The incoming pulse is described by

$$\Phi_{in} = f(Z - T) \exp(iKZ - iWT) \quad (14)$$

where $W = \omega\tau$ and $f(Z - T)$ is the pulse form function, for a Gaussian pulse

$$f^2(Z - T) = \exp[-(Z - T)^2] \quad (15a)$$

and for a hyperbolic secant pulse

$$f^2(Z - T) = \text{sech}^2 (Z - T). \quad (15b)$$

It is convenient to use a new coordinate system which is moving with the pulse, denoting the new coordinates by U_n and V_n where the U_n's and V_n's families are defined by

$$\begin{aligned}U_0 &= Z_0 - T_0 \qquad V_0 = Z_0\\ U_1 &= Z_1 - T_1 \qquad V_1 = Z_1\\ U_2 &= Z_2 - T_2 \qquad V_2 = Z_2 \text{ etc.}\end{aligned} \quad (16)$$

In these variables, (11)–(13) can, respectively, be written as

$$\left[\frac{\partial^2}{\partial V_0^2} + 2 \frac{\partial}{\partial U_0} \frac{\partial}{\partial V_0}\right] \Phi_0 = 0 \quad (11a)$$

$$\begin{aligned}&\left[\frac{\partial^2}{\partial V_0^2} + 2 \frac{\partial}{\partial U_0} \frac{\partial}{\partial V_0}\right] \Phi_1 + 2\left[\frac{\partial}{\partial U_1} \frac{\partial}{\partial V_0} + \frac{\partial}{\partial U_0} \frac{\partial}{\partial V_1}\right.\\ &\left. + \frac{\partial}{\partial V_1} \frac{\partial}{\partial V_0}\right] \Phi_0 = \frac{\partial^2}{\partial U_0^2} |\Phi_0|^2 \Phi_0\end{aligned} \quad (12a)$$

$$\begin{aligned}&\left[\frac{\partial^2}{\partial V_0^2} + 2 \frac{\partial}{\partial U_0} \frac{\partial}{\partial V_0}\right] \Phi_2 + 2\left[\frac{\partial}{\partial U_1} \frac{\partial}{\partial V_0}\right.\\ &\left. + \frac{\partial}{\partial U_0} \frac{\partial}{\partial V_1} + \frac{\partial}{\partial V_1} \frac{\partial}{\partial V_0}\right] \Phi_1\\ &+ \left[\frac{\partial^2}{\partial V_1^2} + 2 \frac{\partial}{\partial U_1} \cdot \frac{\partial}{\partial V_1} + 2 \frac{\partial}{\partial U_0} \cdot \frac{\partial}{\partial V_2}\right.\\ &\left. + 2 \frac{\partial}{\partial U_2} \cdot \frac{\partial}{\partial V_0} + 2 \frac{\partial}{\partial V_2} \cdot \frac{\partial}{\partial V_0}\right] \Phi_0\\ &= 2 \frac{\partial}{\partial U_1} \cdot \frac{\partial}{\partial U_0} |\Phi_0|^2 \Phi_0 + \frac{\partial^2}{\partial U_0^2} |\Phi_1|^2 \Phi_1.\end{aligned} \quad (13a)$$

The above simultaneous differential equations can be solved through the ansatz

$$\begin{aligned}\Phi_0 &= A(U_1, V_1, U_2, V_2)\, e^{iKU_0}\\ \Phi_1 &= 0\\ \Phi_2 &= C(U_1, V_1, U_2, V_2)\, e^{iKU_0}.\end{aligned} \quad (17)$$

It is worth noting at this point that the specific form of C cannot be obtained from the above equations. For cases under consideration, $\epsilon < 1$ and Φ can be adequately approximated by Φ_0.

TABLE II
QUASI-LINEAR PARTIAL DIFFERENTIAL EQUATIONS FORMS FOR THE AMPLITUDE AND PHASE OF A PULSE PROPAGATING IN A CUBIC NONLINEAR MEDIUM

	Amplitude Equation	Phase Equation
Traditional SPM Theory	$\frac{\partial a}{\partial V} = 0$	$\frac{\partial \alpha}{\partial V} = \frac{\epsilon K}{2} a^2$
Slowly Varying Amplitude	$\frac{\partial a}{\partial V} - 3\epsilon a^2 \frac{\partial a}{\partial U} = 0$	$\frac{\partial \alpha}{\partial V} - \epsilon a^2 \frac{\partial \alpha}{\partial U} = \frac{\epsilon K}{2} a^2$
Yang and Shen	$\frac{\partial a}{\partial V} - \frac{\epsilon}{2} a^2 \frac{\partial a}{\partial U} = 0$	$\frac{\partial \alpha}{\partial V} - \frac{\epsilon a^2}{2} \frac{\partial \alpha}{\partial U} = \frac{\epsilon K}{2} a^2$
This Paper	$\frac{\partial a}{\partial V} - \frac{3}{2} \epsilon a^2 \frac{\partial a}{\partial U} = 0$	$\frac{\partial \alpha}{\partial V} - \frac{\epsilon a^2}{2} \frac{\partial \alpha}{\partial U} = \frac{\epsilon K}{2} a^2 - \frac{\epsilon^2 K}{8} a^4$

IV. QUASI-LINEAR FORM FOR THE WAVE EQUATION

In this section, we will specifically derive the quasi-linear form for the wave equation. In effect, we will obtain the system of partial coupled differential equations that relate the pulse amplitude and phase.

Denote

$$A \equiv a\, e^{i\alpha} \tag{18}$$

where a is the amplitude and α is the phase of the pulse; a and α are functions of (U_1, V_1, U_2, V_2).

Substituting (18) and (17) in (11a)–(13a), we obtain

$$\frac{\partial a}{\partial V_1} = 0 \tag{19}$$

$$\frac{\partial \alpha}{\partial V_1} = \frac{K}{2} a^2 \tag{20}$$

$$3a^2 \frac{\partial a}{\partial U_1} = 2 \frac{\partial a}{\partial V_2} \tag{21}$$

$$\frac{K}{4} a^4 + 2 \frac{\partial \alpha}{\partial V_2} = a^2 \frac{\partial \alpha}{\partial U_1}. \tag{22}$$

In the variables U and V, the above equations reduce to

$$\frac{\partial a}{\partial V} - \frac{3}{2} \epsilon\, a^2 \frac{\partial a}{\partial U} = 0 \tag{23}$$

$$\frac{\partial \alpha}{\partial V} - \frac{\epsilon}{2} a^2 \frac{\partial \alpha}{\partial U} = \frac{\epsilon K}{2} a^2 - \frac{\epsilon^2 K}{8} a^4. \tag{24}$$

Equations (23) and (24) are the quasi-linear partial differential equations corresponding to the propagation of a pulse in a nonlinear ($\chi^{(3)}$) medium where dispersion and absorption have been neglected, correct to order ϵ^2.

In the same notation, the quasi-linear partial differential equations previously used are as follows.

A. Traditional Theory

$$\frac{\partial a}{\partial V} = 0 \tag{25}$$

$$\frac{\partial \alpha}{\partial V} = \frac{\epsilon K}{2} a^2. \tag{26}$$

B. Slowly Varying Approximation

$$\frac{\partial a}{\partial V} - 3\epsilon a^2 \frac{\partial a}{\partial U} = 0 \tag{27}$$

$$\frac{\partial \alpha}{\partial V} - \epsilon a^2 \frac{\partial \alpha}{\partial U} = \frac{\epsilon K}{2} a^2. \tag{28}$$

C. Yang and Shen Approximation

$$\frac{\partial a}{\partial V} - \frac{\epsilon}{2} a^2 \frac{\partial a}{\partial U} = 0 \tag{29}$$

$$\frac{\partial \alpha}{\partial V} - \frac{\epsilon a^2}{2} \frac{\partial \alpha}{\partial U} = \frac{\epsilon K}{2} a^2. \tag{30}$$

In all of the above previously used forms for the quasi-linear partial differential equations, assumptions on the rate of change of the amplitude and/or phase were made; however, these assumptions did not have to be invoked in our derivation. Table II summarizes the above results.

V. SOLUTIONS FOR THE AMPLITUDE OF THE PULSE

In this section, specific solutions to (23) will be found for a pulse whose shape function at the input plane is either Gaussian or secant hyperbolic. In what follows, the quantity ϵV will repeatedly occur in the original coordinates $\epsilon V = (n_2 |E_0|^2 / c\tau_L) z$.

A. Gaussian Pulse

The solution [18] of (23) with the boundary condition (15a) is given by

$$a^2 = \exp\left[-(U + \tfrac{3}{2} \epsilon V a^2)^2\right], \tag{31}$$

i.e., the Gaussian pulse is distorted as it propagates. The maximum of the pulse occurs at

$$U_M = -\tfrac{3}{2} \epsilon V \tag{32}$$

and the (amplitude)^2 of the pulse at $U = 0$ is given by the solution of

$$(\epsilon V)^2 = -\frac{4}{9} \frac{\ln a^2(0, V)}{a^4(0, V)}. \tag{33}$$

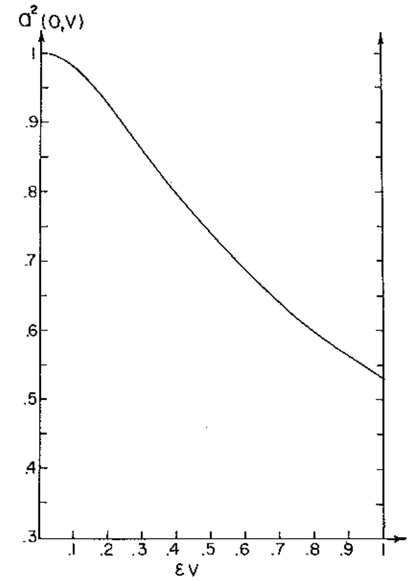

Fig. 1. The (amplitude)^2 of a Gaussian pulse at $U = 0$ as a function of $\epsilon V(= n_2|E_0|^2 z/c\tau_L)$.

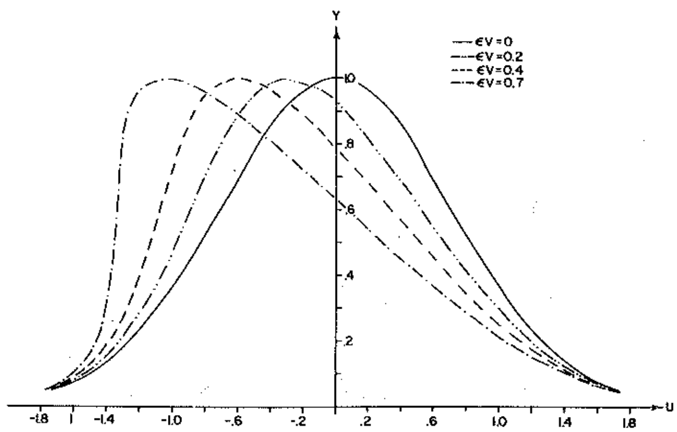

Fig. 2. The development of a Gaussian pulse shape as a function of $\epsilon V (= n_2|E_0|^2 z/c\tau_L)$.

Fig. 1 depicts a^2 (0, V) as a function of ϵV. Fig. 2 shows the development of the Gaussian pulse shape with the variable ϵV. The trail of the pulse steepens as ϵV increases.

This solution obtained under the assumption of neglecting dispersion and absorption is valid as long as no shock wave is formed, i.e., $\partial a^2/\partial U$ finite; this implies that V at the exit plane, denoted by V_e, should satisfy

$$\epsilon V_e < \frac{(2e)^{1/2}}{3} \sim 0.777. \tag{34}$$

For larger values of V_e, the above solution needs to be corrected near the curve sharp edge.

B. Hyperbolic Secant Pulse

The solution of (23) with the boundary condition (15b), i.e., the hyperbolic secant pulse, is given by

$$a^2 = \text{sech}^2 \left[U + \tfrac{3}{2}\, \epsilon V\, a^2\right]. \tag{35}$$

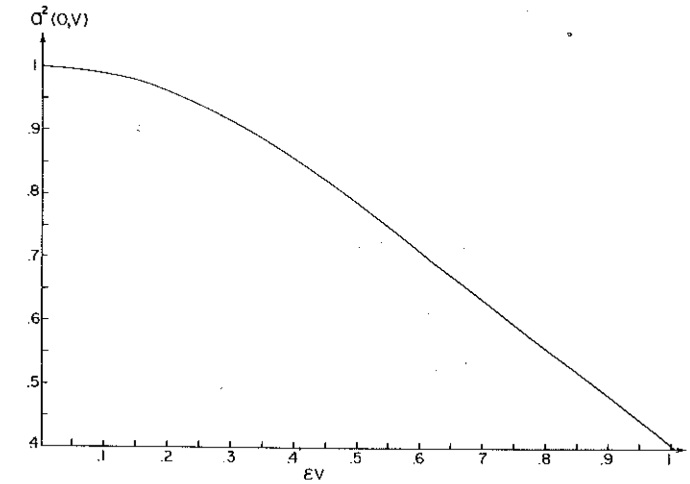

Fig. 3. The (amplitude)^2 of sech pulse at $U = 0$ as a function of $\epsilon V(= n_2|E_0|^2 z/c\tau_L)$.

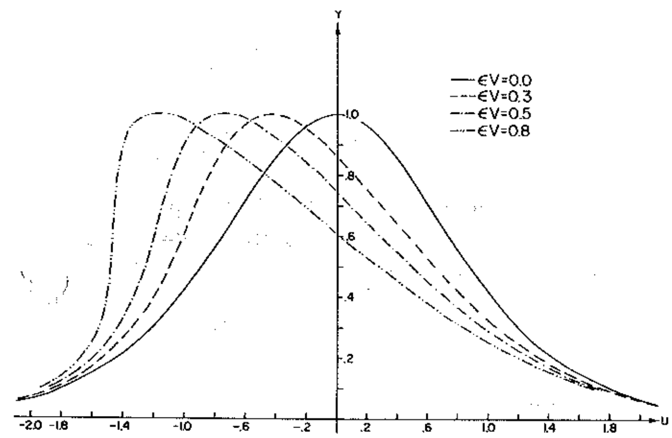

Fig. 4. The development of a sech pulse shape as a function of $\epsilon V(= n_2|E_0|^2 z/c\tau_L)$.

The maximum of the pulse occurs at the same point U_M as that of a Gaussian pulse, and the (amplitude)^2 of the pulse at $U = 0$ is given by the solution of

$$a_0^2 = \text{sech}^2 \left[\tfrac{3}{2}\, \epsilon V\, a_0^2\right] \tag{36a}$$

or

$$\epsilon V = \frac{2}{3a_0^2} \ln \left[\frac{1}{a_0} + \left(\frac{1}{a_0^2} - 1\right)^{1/2}\right] \tag{36b}$$

where $a_0 = a(0, V)$.

Fig. 3 plots a_0^2 as a function of ϵV. Fig. 4 shows the pulse shape for selected values of ϵV for an incoming sech pulse.

The solution given by (36), derived under the conditions of no dispersion and no absorption, is valid if V_e, the value of V at the exit plane, satisfies

$$\epsilon V_e < \frac{\sqrt{3}}{2} \sim 0.866. \tag{37}$$

VI. Spectral Extent

The spectral distribution for the signal at the exit plane is given by

$$S(\omega', z_e) \propto |\tilde{E}(\omega', z_e)|^2 \tag{38}$$

where $\tilde{E}(\omega', z_e)$ is the time Fourier transform of the electric field at the plane $z = z_e$,

$$\tilde{E}(\omega', z_e) = \frac{\mathrm{Re}}{2\pi} \int e^{i\omega' t} E(t, z)\, dt \tag{39}$$

where z_e is the coordinate of the exit plane. In the $U - V$ coordinates, where $W = \omega\tau_L$ and $W' = \omega' t_L$, the expression for $\tilde{E}$ is given by

$$\tilde{E}(W', V_e) = \frac{E_0\tau_L}{2\pi} \mathrm{Re} \int_{-\infty}^{\infty} dU\, e^{i(W-W')U}\, a(U, V_e)\, e^{i\alpha(U, V_e)} \tag{40}$$

where V_e corresponds to the V of the exit plane.

The extent of the spectral distribution can be estimated through the method of stationary phase, specifically,

$$W' - W|_{\text{anti-Stokes}} \simeq \mathrm{Max} \left(\frac{\partial \alpha}{\partial U}\right)_V \tag{41}$$

$$W' - W|_{\text{Stokes}} \simeq \mathrm{Min} \left(\frac{\partial \alpha}{\partial U}\right)_V. \tag{42}$$

The value of $\partial\alpha/\partial U|_V$ is obtained by solving (24).

The general method for solving quasi-linear partial differential equations will be utilized to obtain $\partial\alpha/\partial U|_V$.

Let

$$\alpha' = \alpha + KU; \tag{43}$$

the equation for α' is then given by

$$\frac{\partial \alpha'}{\partial V} - \frac{\epsilon}{2} y \frac{\partial \alpha'}{\partial U} = -\frac{\epsilon^2 K}{8} y^2 \tag{44}$$

where $y = a^2$. The total derivatives [18] form for this equation is then

$$dV = \frac{-2dU}{\epsilon y} = -\frac{8 d\alpha'}{\epsilon^2 K y^2}. \tag{45}$$

From (43)–(45), the general solution for α is given by

$$\alpha = -KU - \frac{K\epsilon}{2} \int_0^U a^2(p, V)\, dp - \frac{3K}{8} \epsilon^2 \int_0^V a^4(0, q)\, dq + F(U^2, V) \tag{46}$$

where the function $F(U, V)$ is a solution of the equation

$$\frac{\partial F}{\partial V} - \frac{\epsilon}{2} a^2 \frac{\partial F}{\partial U} = 0. \tag{47}$$

The boundary condition imposed by the physical condition is that at the input plane (i.e., $V = 0$), α should be zero for all values of U. This implies that, for an incoming sech pulse,

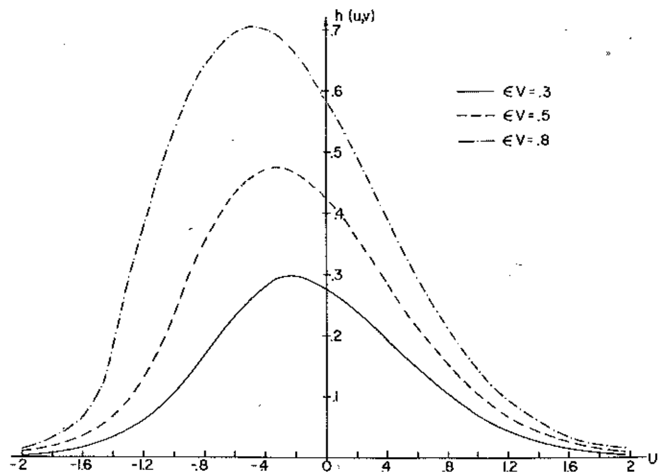

Fig. 5. The function $h(U, V) = \epsilon \int_0^V a^3(U, q)\, dq$ for a sech pulse.

$$F(U, 0) = KU + \frac{K\epsilon}{2} \tanh U. \tag{48}$$

The general solution of (47) is given by

$$F(f) = F\left(\frac{\epsilon}{2} \int_0^V a^3(U, q)\, dq + \int_0^U a(p, 0)\, dp\right). \tag{49}$$

The special form for α satisfying the constraint (48) is given by

$$\alpha = -KU - \frac{K\epsilon}{2} \int_0^U a^2(p, V)\, dp - \frac{3K}{8} \epsilon^2 \int_0^V a^4(0, q)\, dq + K \tanh^{-1} \sin f(U, V) + \frac{K\epsilon}{2} \sin f(U, V) \tag{50}$$

where

$$f(U, V) = \tfrac{1}{2} h(U, V) + \sin^{-1} \tanh U \tag{51a}$$

and

$$h(U, V) = \epsilon \int_0^V a^3(U, q)\, dq.$$

In Fig. 5, $h(U, V)$ is plotted as a function of U for different values of ϵV.

The expression for the phase change $\partial\alpha/\partial U$ is then given by

$$\frac{\partial \alpha}{\partial U} = -K - \frac{K\epsilon}{2} a^2(U, V) + K a(U, V) [\cos f(U, V)]^{-1} + \frac{K\epsilon}{2} a(U, V) \cos f(U, V). \tag{52}$$

In Fig. 6, the instantaneous relative frequency sweep $(1/K)\,(\partial\alpha/\partial U)$ is plotted for different values of ϵV. Significant changes occur close to the trailing edge. In Fig. 7, the Stokes and anti-Stokes spectral extents are plotted as

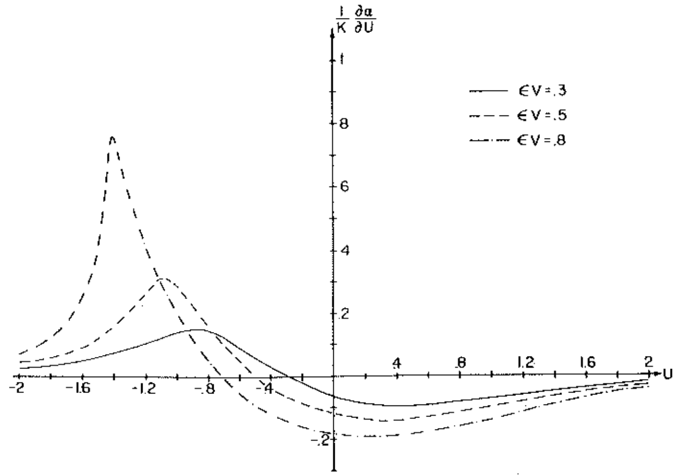

Fig. 6. The derivative of the phase for a sech pulse.

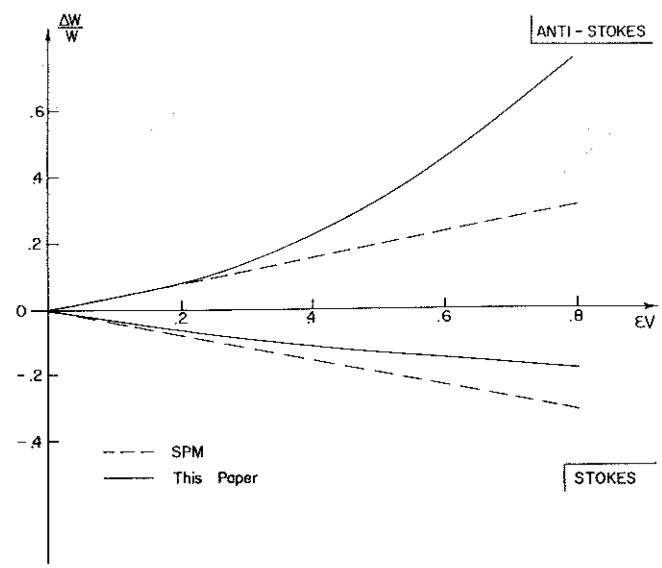

Fig. 7. The Stokes and anti-Stokes spectral distribution extents as a function of ϵV for a sech pulse ($\epsilon \ll 1$).

functions of ϵV for this calculation and for traditional SPM. For $\epsilon V \ll 1$, this calculation and traditional SPM coincide as they should. For larger values of ϵV, the asymmetry between Stokes and anti-Stokes sides is substantial.

VII. Conclusion

In this paper, we derived the quasi-linear system of partial differential equations that describe the propagation of a pulse in a nonlinear cubic medium. From the same equations, we were able to derive the results pertaining to the pulse distortion and to the spectral distribution of the supercontinuum. In particular, the asymmetries, observed experimentally, in both physical quantities, are obtained as a result of propagation in the medium. Furthermore, we were able to relate the two physical quantities, i.e., the pulse shape and the spectral distribution of a supercontinuum, in a functional form. Equation (52) permits experimental results on pulse distortion and the spectral distribution of the supercontinuum to be interrelated. Finally, it is worth emphasizing that the above phenomenological theory predicts, within its domain of validity, the range of experimentally observed spectral extents, without recourse to any additional assumptions.

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